Section 3.2.1 – Arithmetic Sequence

VCAA "Dot Points"

Depreciation of assets, including:

- review of the use of a first-order linear recurrence relation to generate the terms of a sequence
- use of a recurrence relation to model and compare (numerically and graphically) flat rate, unit cost and reducing balance depreciation of the value of an asset with time, including the use of a recurrence relation to determine the depreciating value of an asset after n depreciation periods, including from first principles for n ≤ 5
- use of the rules for the future value of an asset after n depreciation periods for flat rate, unit cost and reducing balance depreciation and their application.

Arithmetic Sequence

The following examples are classified as **arithmetic sequences**. Each term in an arithmetic sequence, have a **common difference**.

Example 1

Write a recursion relation for the following sequence:

2, 4, 6, 8, 10, 12 …

$$2 \xrightarrow{+2} 4 \xrightarrow{+2} 6 \xrightarrow{+2} 8 \xrightarrow{+2} 10 \xrightarrow{+2} 12$$

Initial value: 2

Description: "To generate successive terms, add 2 to the previous term"

Written as a recursion relation:

$$\underbrace{U_{n+1}}_{\text{Next term}} = \overbrace{U_n}^{\text{Current term}} + \underbrace{2}_{\text{Common difference}}; \quad \overbrace{U_0}^{\text{Initial term}} = 2$$

Example 2

Write a recursion relation for the following sequence:

50, 40, 30, 20, 10 …

$$50 \xrightarrow{-10} 40 \xrightarrow{-10} 30 \xrightarrow{-10} 20 \xrightarrow{-10} 10$$

Initial value: 50

Description: "To generate successive terms, take 10 from the previous term"

Written as a recursion relation:

$$\underbrace{U_{n+1}}_{\text{Next term}} = \overbrace{U_n}^{\text{Current term}} - \underbrace{10}_{\text{Common difference}}; \quad \overbrace{U_0}^{\text{Initial term}} = 50$$

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Example 3

Construct the first five terms for following the recursion relation:

$$\underset{\text{Next term}}{\boldsymbol{U_{n+1}}} = \overset{\text{Current term}}{\boldsymbol{U_n}} + \underset{\text{Common difference}}{\boldsymbol{4}}; \quad \boldsymbol{U_0} = \overset{\text{Initial term}}{\boldsymbol{10}}$$

Initial value: 10
Description: "Add 4 to the previous value"

10, 14, 18, 22, 26

$$10 \xrightarrow{+4} 14 \xrightarrow{+4} 18 \xrightarrow{+4} 22 \xrightarrow{+4} 26$$

Example 4

Construct the first five terms for following the recursion relation:

$$\underset{\text{Next term}}{\boldsymbol{U_{n+1}}} = \overset{\text{Current term}}{\boldsymbol{U_n}} - \underset{\text{Common difference}}{\boldsymbol{5}}; \quad \boldsymbol{U_0} = \overset{\text{Initial term}}{\boldsymbol{25}}$$

Initial value: 25
Description: "Take 5 from the previous value"

25, 20, 15, 10, 5

$$25 \xrightarrow{-5} 20 \xrightarrow{-5} 15 \xrightarrow{-5} 10 \xrightarrow{-5} 5$$

Applications of Arithmetic Sequence

Arithmetic sequences have several applications in the field of **financial modelling**. Let's examine these in a summarized format:

$$U_{n+1} = U_n \boxed{\pm d}$$

Appreciation (+)

When the difference (d) is a **positive value**

$$U_{n+1} = U_n + d$$

Appreciation involves the **increase** of the value of an initial investment or item, via a common difference (D) every period of investment.

Depreciation (-)

When the difference (d) is a **negative value**

$$U_{n+1} = U_n - d$$

Depreciation involves the **decrease** of the value of an initial investment or item, via a common difference (D) every period of investment.

Application: Simple Interest

$$d = \frac{r}{100} \times V_0$$

In the case of **simple interest**, a fixed amount of money is added to the value of an investment every term/period.

This added amount is usually expressed as a percentage of the principal or initial value.

Application: Flat Rate Depreciation

$$d = \frac{r}{100} \times V_0$$

In the case of **flat rate depreciation**, a fixed amount of money is deducted from the value of an item every term/period.

This deducted amount is usually expressed as a percentage of the initial value.

Application: Unit Cost Depreciation

In the case of **unit cost depreciation**, a fixed amount of money is deducted from the value of an item every unit of use, such as per kilometer, copy etc.

Application: Simple Interest

$\boldsymbol{U_{n+1} = U_n + d}$

Whether you borrow or invest money, you will expect to either pay or earn **interest**. The formula used to calculate **simple** or **flat rate interest** is as follows:

$$I = \frac{V_0Rn}{100}$$

$$V_0 = \frac{100I}{Rn}$$

$$R = \frac{100I}{V_0n}$$

$$n = \frac{100I}{V_0R}$$

Where I = simple interest charged or earned ($)
V_0 = principal ($)
R = rate of interest per period (% per period)
n = number of periods of agreement

NB: The period is typically expressed in years, months, quarters, fortnights, weeks or days.

As simple interest increases the value of the investment by the same amount each period, it can be considered as a **recursion model**.

Accordingly, simple interest can be expressed via the following relation/rule:

$$V_{n+1} = V_n + d\ ,\ d = \frac{V_0R}{100}$$

Where V_n represents the value of the investment ($)
n represents the time periods
d represents the interest earned ($) per period
V_0 = principal ($) or starting amount
R = interest rate

At the end of the term, the **total amount** is equal to the principal plus the interest

$$V_n = V_0 + I$$

Example.5

Consider an investment of \$1000 earning simple interest at a rate 5% p.a. over a period of 10 years.

Task.1 Set up a recurrence relation to find the value of the investment after n years.

Task.2 Create an investment schedule for the 10 years duration of the investment.

Task.3 Construct a graph of the investment value versus periods in years.

Task.4 Calculate the total interest earned and the total amount the investment is worth at the end of the term.

Task.1 Set up a recurrence relation to find the value of the investment after n years.

d = ? $\quad d = \frac{V_0 R}{100}$

V_0 = \$1000 $\quad = \frac{1000 \times 5}{100}$

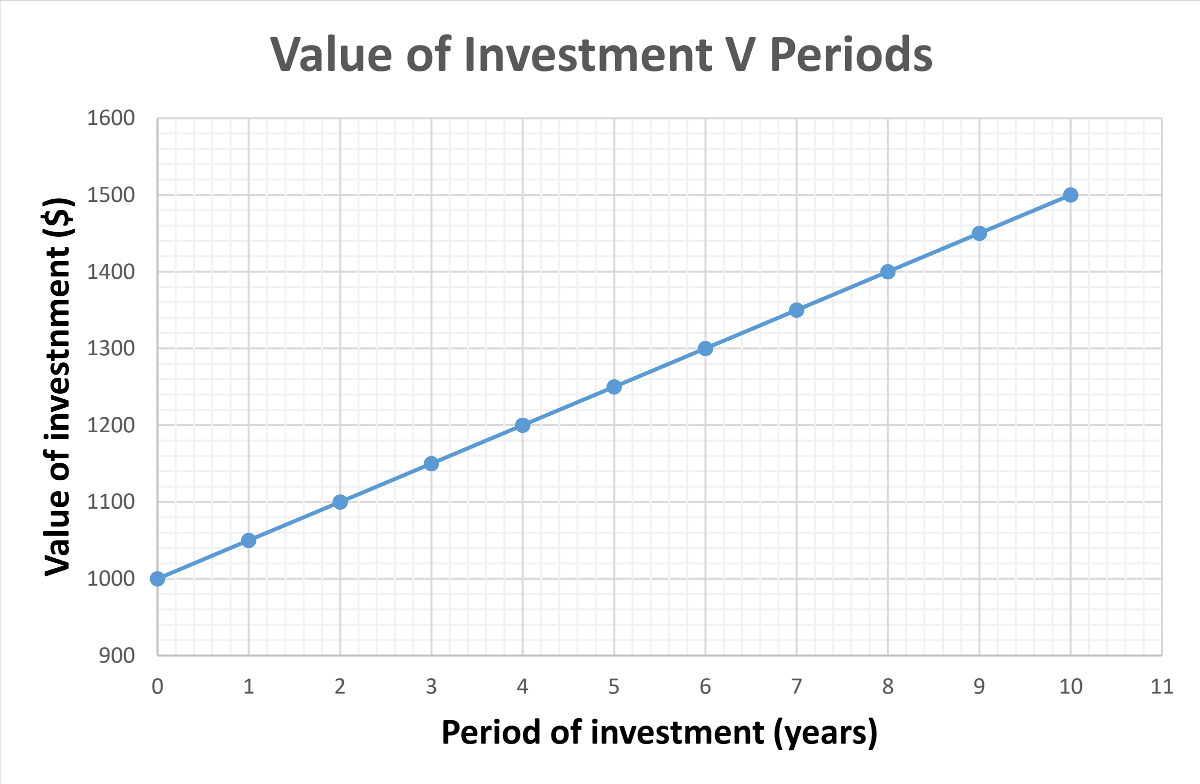
R = 5% p.a. $\quad$ = <u>\$50.00</u>

$\therefore V_{n+1} = V_n + 50$

Task.2 Create an investment schedule for the 10 years duration of the investment.

n (years)	Starting value (\$)	Interest (\$)	Amount (\$)
0	1000.00	0.00	1000.00
1	1000.00	50	1050.00
2	1050.00	50	1100.00
3	1100.00	50	1150.00
4	1150.00	50	1200.00
5	1200.00	50	1250.00
6	1250.00	50	1300.00
7	1300.00	50	1350.00
8	1350.00	50	1400.00
9	1400.00	50	1450.00
10	1450.00	50	1500.00

Task.3 Construct a graph of the investment value versus periods in years.



Task.4 Calculate the total interest earned and the total amount the investment is worth at the end of the term.

Step.1 calculate the total interest earned after 10 years.

I = ?
V_0 = $1000.00
R = 5% p.a.
n = 10 years

$$I = \frac{V_0 R n}{100}$$

$$= \frac{1000 \times 5 \times 10}{100}$$

$$= \$500$$

Step.2 calculate the total value (amount) of the investment after 10 years.

$\therefore V_n = V_0 + I$
= 1000.00 +500.00
= **$1500.00**

Example.6

Lisa invests \$4,200 for 9 months at an interest rate of 5.2% p.a.
Calculate her interest earned and the total amount at the end of the term.

Step.1

I = ?
V_0 = \$4,200
R = 5.2% p.a.
n = 9 months
= $\frac{9}{12}$ or 0.75 yrs

$$I = \frac{V_0 R n}{100}$$

$$= \frac{4200 \times 5.2 \times 0.75}{100}$$

= **<u>\$163.80</u>**

NB: The period of the interest rate and that of the terms must be the same.

Eg. Rate % p.a. (**per annum**) and n in **years**, or;
Rate % p.**month** and n in **months**, or;
Rate % p.**quarter** and n in **quarters**

In this example both the interest rate (R) and number of periods (n) are measured in months.

Step.2

$$V_n = V_0 + I$$

= 4200.00 + 163.80
= **<u>\$4363.80</u>**

So, Lisa would earn \$163.80 interest and the total amount at the end of the term would be \$4,363.80.

Example.7

Jack has invested \$5000 in a term deposit. How long, to the nearest month, will it take for him to earn \$740 simple interest, if the rate is 6.4% p.a.?

n = ?
I = \$740
V_0 = \$5000
R = 6.4% p.a.

NB: The period of the interest rate and that of the terms must be the same.

As the rate has been entered into the equation in % p.a. The periods (n) is calculated in terms of years.

$$n = \frac{100I}{V_0 R}$$

$$n = \frac{100 \times 740}{5000 \times 6.4}$$

= 2.3125 years

= 2 Yrs and (0.3125 x 12) months

= 2 Yrs and 3.75 months

≈ 2 Yrs and 4 months

Application: Flat rate depreciation
$\boldsymbol{U_{n+1} = U_n - d}$

Flat rate depreciation, also known as **straight line depreciation**, is when an item depreciates by a fixed amount each unit if time, commonly per year.

Depreciation can be expressed in dollars ($) or as a percentage of the cost price(%).

Flat rate depreciation can be modelled using a recursion rule:

$$\boxed{V_{n+1} = V_n - d}$$

Where V_n is the value of the asset ($) after n depreciation periods
d is the depreciation in each time period.

The following equation can be used to calculate the future value of a depreciating item:

$$\boxed{V_n = V_0 - nd}$$

Where n represents the number of depreciation periods.

Example.8
James purchases a new car valued at $50 000. For taxation purposes James chooses to depreciate his car by the **flat rate method**. The depreciation was 10% of the cost price and its useful life was 8 years.

1. Find the annual depreciation
2. Draw a depreciation schedule for the useful life of the car
3. Construct a graph of future value against time
4. Find the relationship between the future value and time and
5. Use it to find the scrap value

Part.1

V_0 = \$50 000
d = 10% of cost price
= 10% of \$50 000
= \$5 000

Annual depreciation is \$5 000

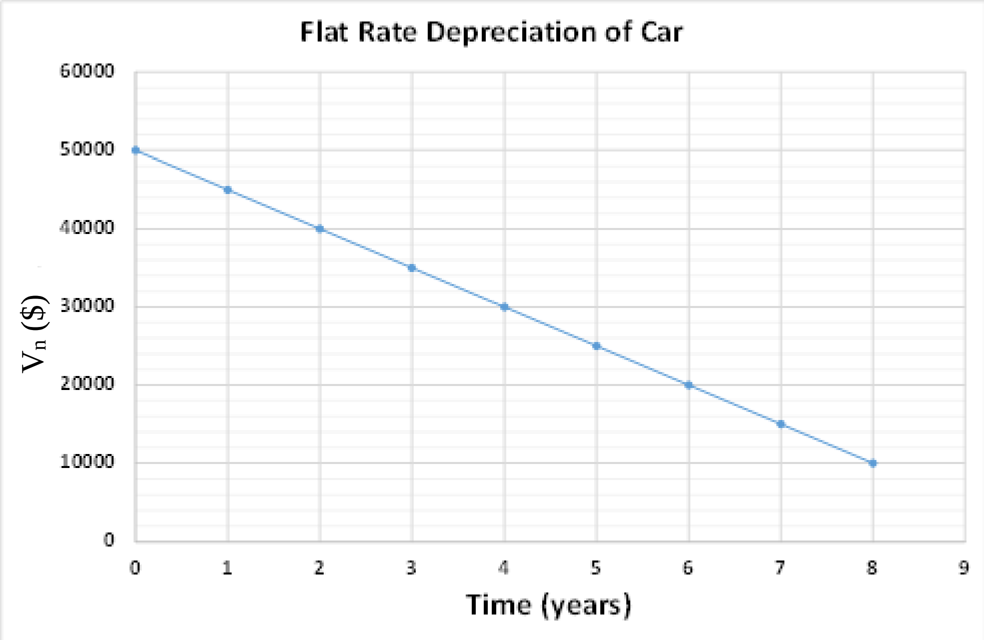
Part.2

Depreciation schedule for 8 years, using depreciation of \$5000 each year, starting at a cost price of \$50000

T (years)	d (\$)	V_n (\$)
0		50000
1	5000	45000
2	5000	40000
3	5000	35000
4	5000	30000
5	5000	25000
6	5000	20000
7	5000	15000
8	5000	10000

Part.3

Graph showing future value (V_n) against time



Part.4

Construct the relationship between V_n and T

d = \$5000
V_0 = \$50000

Substitute into the equation:

$$V_n = V_0 - nd$$
$$\therefore\ V_n = 50000 - 5000n \text{ or}$$
$$V_n = -5000n + 50000$$

NB: This equation also follows that of a straight line graph, $y = a + bx$
Where y represents the book value V_n
a represents the starting, or rather cost price
b represents the rate of depreciation

Part.5

Calculate the scrap value after 8 years.
Substitute n = 8 into the above equation

$V_n = 50000 - 5000n$
$V_8 = 50000 – 5000(8)$
$= 50000 – 40000$
$= \underline{\$10000}$

The scrap value after 8 years is $10000

Example.9

Naomi bought her car 5 years ago for $5000. Its current market value is $750. Assuming straight line depreciation:

1. Find the car's annual depreciation rate
2. Find the relationship between the future value and time and
3. Use it to find when the car will have a value of $325

Part.1

Total depreciation = cost price - current value
= $5000 - $750
= $4250

$$Rate\ of\ depreciation = \frac{Total\ depreciation}{No\ of\ years}$$

$$= \frac{\$4250}{5\ years}$$

$$= \$850\ per\ year$$

The annual depreciation rate is $850

Part.2

Create a future value equation.

V_0 = \$5000
d = \$850

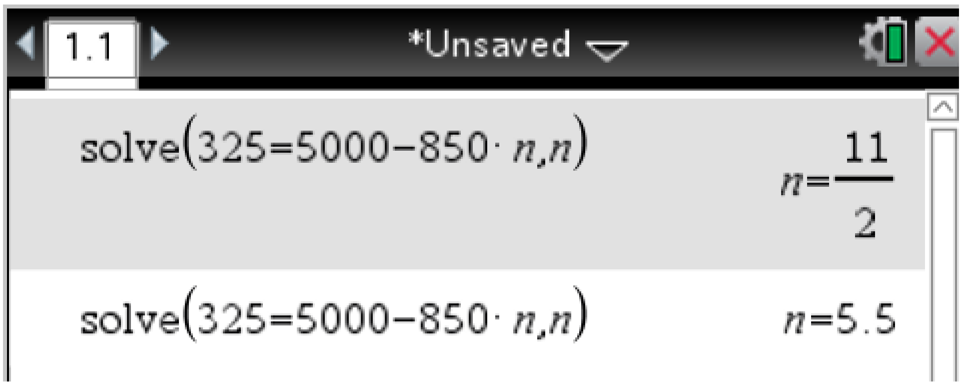
Substitute into the equation:

$$V_n = V_0 - nd$$
$$\therefore\ V_n = 5000 - 850n \text{ or}$$
$$V_n = -850n + 5000$$

Part.3

Substitute V_n = \$325 into the above equation to solve for n

$$V_n = 5000 - 850n$$
$$325 = 5000 - 850n$$



The car will reach a book value of \$325 after 5.5 years

Application: Unit Cost Depreciation

$\boldsymbol{U_{n+1} = U_n - D}$

Both flat rate depreciation and reducing balance depreciation are subject to the age of the item. That is the depreciation is **time based**.

In the case of **unit cost depreciation**, the depreciation is based upon the maximum output (units) of the item.

For example;

- the life of a car is considered in kilometres travelled rather than years of age.
- The life of a printer is considered in pages printed rather than years of age.

This makes sense as some people drive their car 100km or more a day to get to and from work and accumulate a large number of kilometres travelled annually. Whereas, others will drive a car down the street once or twice a week, accumulating very few kilometres annually.

Unit cost depreciation can be modelled using a recursion rule:

$$\boxed{V_{n+1} = V_n - d}$$

Where V_n is the value of the asset ($) after n outputs
d is the depreciation per output

NB: Outputs could be per kilometre, per hour, per copy

The following equation can be used to calculate the future value of a depreciating item:

$$\boxed{V_n = V_0 - nd}$$

Where n represents the number of outputs

Other useful equations:

$$\boxed{\text{Amount of depreciation} = \text{amount of use} \times \text{rate}}$$

or alternatively;

$$\boxed{Rate\ of\ depreciation = \frac{Amount\ of\ depreciation}{Amount\ of\ use}}$$

Example.10

A taxi was purchased for $45000 and depreciates by 25 cents per km driven. During its first year the taxi travelled 78000 km.

Calculate:

1. The depreciation of the taxi in its first year
2. The value of the taxi after 100000 km travelled
3. How far it had travelled if its total depreciation was $30000

Part 1

Amount of depreciation = ?
Amount used = 78000 km
Rate = $0.25 per km

Amount of depreciation = amount of use $\times$ rate
= 78000×0.25
= **<u>$19500</u>**

After its first year (travelling a distance of 78000 km) the taxi depreciated $19500.

Part 2

V_{100000} = ?
V_0 = $45000
d = $0.25 per km
n = 100000 km

$V_{100000} = V_0 - nd$
= $45000 - (100000 \times 0.25)$
= **<u>$20000</u>**

After it's the taxi has travelling a distance of 100000 km the taxi has a value of $20000.

Part 3

Amount used = ?
Amount of depreciation = $30000
Rate = $0.25 per km

Amount of depreciation = amount of use $\times$ rate

$$\therefore Amount\ of\ use = \frac{Amount\ of\ depreciation}{Rate}$$

$$= \frac{\$30\,000}{\$0.25\ per\ km}$$

= **<u>120 000 km</u>**

The taxi would need to travel 120,000 km to depreciate by $30,000.

Example.11

A machine which was purchased for $10,000 was depreciated at the rate of 5 cents per unit produced. What would be the number of units produced, if the machine's book value (future value) had decreased to $4000?

V_n = $4000
V_0 = $10000
d = $0.05 per unit produced
n = ?

$V_n = V_0 - nd$
$4000 = 10000 - [n \times 0.05]$

$$\text{solve}(4000=10000-n\cdot 0.05,n)$$

$$n=120000.$$

The machine's book value will decrease to $4000 after 120,000 units have been produced by the machine.

Example.12

A machine which was purchased for $10,000 was depreciated at the rate of 5 cents per unit produced. What would be the number of units produced, if the machine's book value (future value) had decreased by $8000?

Amount of use = ?
Amount of depreciation = $8000
Rate = $0.05 per unit produced

Amount of depreciation = amount of use ×rate

$$\therefore Amount\ of\ use = \frac{Amount\ of\ depreciation}{Rate}$$

$$= \frac{\$8000}{\$0.05\ per\ unit}$$

= 160000 units produced

The machine's book value will decrease by $8000 after 160,000 units have been produced by the machine.

Exam Styled Questions – Multiple Choice

Question 1
(2017 VCAA Exam 1 Section A - Qn 18)

The first five terms of a sequence are 2, 6, 22, 86, 342 …
The recurrence relation that generates this sequence could be

A. $P_0 = 2$, $P_{n+1} = P_n + 4$
B. $P_0 = 2$, $P_{n+1} = 2P_n + 2$
C. $P_0 = 2$, $P_{n+1} = 3P_n$
D. $P_0 = 2$, $P_{n+1} = 4P_n - 2$
E. $P_0 = 2$, $P_{n+1} = 5P_n - 4$

2	2.
2.· 4−2	6.
6.· 4−2	22.
22.· 4−2	86.
86.· 4−2	342.

D

Question 2
(2016 VCAA Exam 1 Section A - Qn 17)

Consider the recurrence relation below.
$A_0 = 2$, $A_{n+1} = 3A_n + 1$

The first four terms of this recurrence relation are

A. 0, 2, 7, 22 …
B. 1, 2, 7, 22 …
C. 2, 5, 16, 49 …
D. 2, 7, 18, 54 …
E. 2, 7, 22, 67 …

2	2.
2.· 3+1	7.
7.· 3+1	22.
22.· 3+1	67.

E

Question 3
(2016 VCAA Sample Exam 1 Section A - Qn 17)

$$P_0 = 2000,\ P_{n+1} = 1.5P_n - 500$$

The first three terms of a sequence generated by the recurrence relation above are

A. 500, 2500, 2000 …
B. 2000, 1500, 1000 …
C. 2000, 2500, 3000 …
D. 2000, 2500, 3250 …
E. 2000, 3000, 4500 …

2000	2000.
2000.· 1.5−500	2500.
2500.· 1.5−500	3250.

D

Exam Styled Questions – Short Answer

Question 1 (5 marks)
(2017 VCAA Exam 2 Section A - Qn 5)

Alex is a mobile mechanic.
He uses a van to travel to his customers to repair their cars.
The value of Alex's van is depreciated using the flat rate method of depreciation.
The value of the van, in dollars, after *n* years, V_n , can be modelled by the recurrence relation shown below.

$$V_0 = 75\,000,\ V_{n+1} = V_n - 3375$$

a. Recursion can be used to calculate the value of the van after two years.
Complete the calculations below by writing the appropriate numbers in the boxes provided.

$V_0 =$ 75 000

$V_1 =$ 75 000 – $\boxed{3375}$ = 71 625

$V_2 =$ $\boxed{71\,625}$ – $\boxed{3375}$ = $\boxed{68\,250}$

2 marks

b. i. By how many dollars is the value of the van depreciated each year?
The van is depreciating $3375 each year.

1 mark

ii. Calculate the annual flat rate of depreciation in the value of the van.
Write your answer as a percentage.

$r = ?$
$d = \$3375$
$V_0 = \$75000$

$$d = \frac{r}{100} \times V_0$$

$$\therefore r = \frac{d}{V_0} \times 100$$

$$r = \frac{3375}{75000} \times 100 = 4.5\ \%\ p.a.$$

1 mark

Question 2 (3 marks)
(2016 VCAA Sample Exam 2 Section A - Qn 6)

Ken's first caravan had a purchase price of $38 000.
After eight years, the value of the caravan was $16 000.

a. Show that the average depreciation in the value of the caravan per year was $2750.

V_0 = $38000
V_8 = $16000
n = 8
d = ?

$$V_n = V_0 - nd$$
$$V_8 = V_0 - 8d$$
$$16000 = 38000 - 8d$$

solve(16000=38000−8· d,d) d=2750.

∴ the average depreciation is equal to $2750 per year.

1 mark

b. Let C_n be the value of the caravan *n* years after it was purchased.
Assume that the value of the caravan has been depreciated using the **flat rate** method of depreciation.
Write down a recurrence relation, in terms of C_{n+1} and C_n that models the value of the caravan.

C_0 = *38000*, $C_{n+1} = C_n - 2750$

1 mark

c. The caravan has travelled an average of 5000 km in each of the eight years since it was purchased.
Assume that the value of the caravan has been depreciated using the **unit cost** method of depreciation.
By how much is the value of the caravan reduced per kilometre travelled?

V_n = $16000
V_0 = $38000
d = ? $/km
n = 5000 km × 8
= 40000 km

$$V_n = V_0 - nd$$
$$V_8 = V_0 - 8d$$
$$16000 = 38000 - 40000d$$

solve(16000=38000−40000· d,d) d=0.55

The value of the caravan is being reduced by 55 cents every kilometre travelled

1 mark