Section 3.2.2 – Geometric Sequence

VCAA "Dot Points"

Depreciation of assets, including:

- review of the use of a first-order linear recurrence relation to generate the terms of a sequence
- use of a recurrence relation to model (numerically and graphically) reducing balance depreciation of the value of an asset with time, including the use of a recurrence relation to determine the depreciating value of an asset after n depreciation periods, including from first principles for $n \leq 5$
- use of the rules for the future value of an asset after n depreciation periods for flat rate, unit cost and reducing balance depreciation and their application.

Compound interest investments and loans, including:

- review of the concepts of simple and compound interest
- use of a recurrence relation to model and analyse (numerically and graphically) a compound interest investment or loan, including the use of a recurrence relation to determine the value of the compound interest loan or investment after n compounding periods, including from first principles for $n \leq 5$
- rule for the future value of a compound interest investment or loan after n compounding periods and its use to solve practical problems.

Reducing balance loans (compound interest loans with periodic repayments), including:

- the use of a recurrence relation to determine the value of the loan or investment after n payments, including from first principles for $n \leq 5$

Geometric Sequence

The following examples are classified as **geometric sequences**. Each term in a geometric sequence, have a **common factor or ratio**.

Example 1

Write a recursion relation for the following sequence:
2, 4, 8, 16, 32, 64 …

$$2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{\times 2} 32 \xrightarrow{\times 2} 64$$

Initial value: 2
Description: "To generate successive terms, multiply 2 to the previous term"

Written as a recursion relation:

Current term *Initial term*

$$U_{n+1} = 2U_n; \quad U_0 = 2$$

Next term *Common factor*

Example 2

Write a recursion relation for the following sequence:
625, 125, 25, 5, 1 …

$$625 \xrightarrow{\times \frac{1}{5}} 125 \xrightarrow{\times \frac{1}{5}} 25 \xrightarrow{\times \frac{1}{5}} 5 \xrightarrow{\times \frac{1}{5}} 1$$

Initial value: 625
Description: "To generate successive terms, divide the previous term by 5"

Written as a recursion relation:

Current term *Initial term*

$$U_{n+1} = \frac{1}{5}U_n; \quad U_0 = 625$$

Next term *Common factor*

Or alternatively

$$U_{n+1} = 0.2U_n; \quad U_0 = 625$$

Example 3

Construct the first five terms for following the recursion relation:

Current term *Initial term*

$$U_{n+1} = 3U_n; \quad U_0 = 1$$

Next term *Common factor*

Initial value: 1
Description: "Multiple to the previous value by 3"

1, 3, 9, 27, 81

$$1 \xrightarrow{\times 3} 3 \xrightarrow{\times 3} 9 \xrightarrow{\times 3} 27 \xrightarrow{\times 3} 81$$

Example 4

Construct the first five terms for following the recursion relation:

Current term *Initial term*

$$U_{n+1} = 0.5U_n; \quad U_0 = 64$$

Next term *Common factor*

Initial value: 64
Description: "Divide the previous value by 2"

64, 32, 16, 8, 4

Applications of Geometric Sequence

Geometric sequences have several applications in the field of **financial modelling**. Let's examine these in a summarized format:

$$U_{n+1} = \boxed{R}U_n$$

Geometric Growth (R>1)

When the factor **(R) > 1** the next term will **increase** or **grow**.

$$U_{n+1} = RU_n$$
Eg. $U_{n+1} = 1.2U_n$

Application: Compounding interest (investments and loans)

Recurrence relation:
$V_{n+1} = (1 + \frac{r}{100}) \times V_n, V_0 = principal$

Equation for the n^th term:
$$V_n = (1 + \frac{r}{100})^n \times V_0$$

Where:
n = number of compounding periods
V_n = value after n compounding periods
r = the interest rate (%) per compounding period

Geometric Decay (R<1)

When the factor **(R) < 1** the next term will **decrease** or **decay**.

$$U_{n+1} = RU_n$$
Eg. $U_{n+1} = 0.8U_n$

Application: Reducing balance depreciation

Recurrence relation:
$V_{n+1} = (1 - \frac{r}{100}) \times V_n, V_0 = principal$

Equation for the n^th term:
$$V_n = (1 - \frac{r}{100})^n \times V_0$$

Where:
n = number of compounding periods
V_n = value after n compounding periods
r = the interest rate (%) per compounding period

Application.1: Compounding Interest
$U_{n+1} = RU_n$, **where R > 1**

In the case of ***simple interest***, interest is only paid at the end of a specified term. The same amount of interest is paid into your account every term based upon the principal and interest rate.

Example.1
If you invested $10,000 at 5% per year, you would earn $2,500 in simple interest after 5 years, **$500 for each year**. This would give you a total of $12,500 after 5 years.

Simple interest on a $1000 investment at 5% per year

	Year 1	Year 2	Year 3	Year 4	Year 5
Deposit	$1000	$0	$0	$0	$0
Interest	$50	$50	$50	$50	$50
Total	$1050	$1100	$1150	$1200	$1250

However, with ***compound interest*** the interest earned in the first time period is added to the total and then the interest for the next period is calculated. You are actually earning interest on your interest.

Compound interest on a $1000 investment at 5%p.a (compounding annually)

	Year 1	Year 2	Year 3	Year 4	Year 5
Deposit	$1000	$0	$0	$0	$0
Interest	$50.00	$52.50	$55.13	$57.88	$60.77
Total	$1050.00	$1102.50	$1157.63	$1215.51	1276.28

NB: The examples above demonstrates how the interest earned using simple interest remains the same each year and how the interest earned using compound interest grows every year. You will earn more money if you are paid compounding interest.

Compounding interest can also be considered as a **recursion relation**. However, instead of being a recurrence relation with a ***common difference***. Compounding interest has a recurrence relation with a ***common factor*** or ***ratio***.

Accordingly, compound interest can be expressed via the following relation rule:

$$\boxed{V_{n+1} = V_n R}$$

Where V_{n+1} represents the value of the investment 1 time period after V_n ($)
R represents the growth or compounding factor (= $1 + \frac{r}{100}$)

Or in a more generic form:

$$\boxed{V_n = V_0 R^n}$$

Where V_n represents the final or total amount ($)
V_0 represents the principal ($)
R represents the growth or compounding factor (= $1 + \frac{r}{100}$)
r represents the interest rate per period
n represents the number of interest-bearing periods

At the end of the term, the interest earned can be easily calculated using the following equation:

$$\boxed{I = V_n - V_0}$$

Understanding Compound Interest

Consider the previous compounding interest example where $1000 was invested at 5%p.a (compounding annually). Let's examine the compounding nature of compounding interest.

NB: at an interest rate of 5% p.a. compounded, the growth factor is **1.05** (= $1 + \frac{5}{100}$)

Year 1

$V_1 = V_0R$
$= 1000 \times \mathbf{1.05}$
$= \underline{\$1050}$

Year 2

$V_2 = V_1R$
$= 1050 \times 1.05$
$= \underline{\$1102.50}$

or

$V_2 = V_0R^2$
$= (1000) \times \mathbf{1.05} \times \mathbf{1.05}$
$= 1000 \times (\mathbf{1.05})^2$
$= \underline{\$1102.50}$

Year 3

$V_3 = V_2R$
$= 1102.50 \times 1.05$
$= \underline{\$1157.63}$

or

$V_3 = V_0R^3$
$= (1000) \times \mathbf{1.05} \times \mathbf{1.05} \times \mathbf{1.05}$
$= 1000 \times (\mathbf{1.05})^3$
$= \underline{\$1157.63}$

Year 4

$V_4 = V_3R$
$= 1157.625 \times 1.05$
$= \underline{\$1215.51}$

or

$V_4 = V_0R^4$
$= (1000) \times \mathbf{1.05} \times \mathbf{1.05} \times \mathbf{1.05} \times \mathbf{1.05}$
$= 1000 \times (\mathbf{1.05})^4$
$= \underline{\$1215.51}$

Year 5

$V_5 = V_4R$
$= 1215.50625 \times 1.05$
$= \underline{\$1276.28}$

or

$V_5 = V_0R^5$
$= (1000) \times \mathbf{1.05} \times \mathbf{1.05} \times \mathbf{1.05} \times \mathbf{1.05} \times \mathbf{1.05}$
$= 1000 \times (\mathbf{1.05})^5$
$= \underline{\$1276.28}$

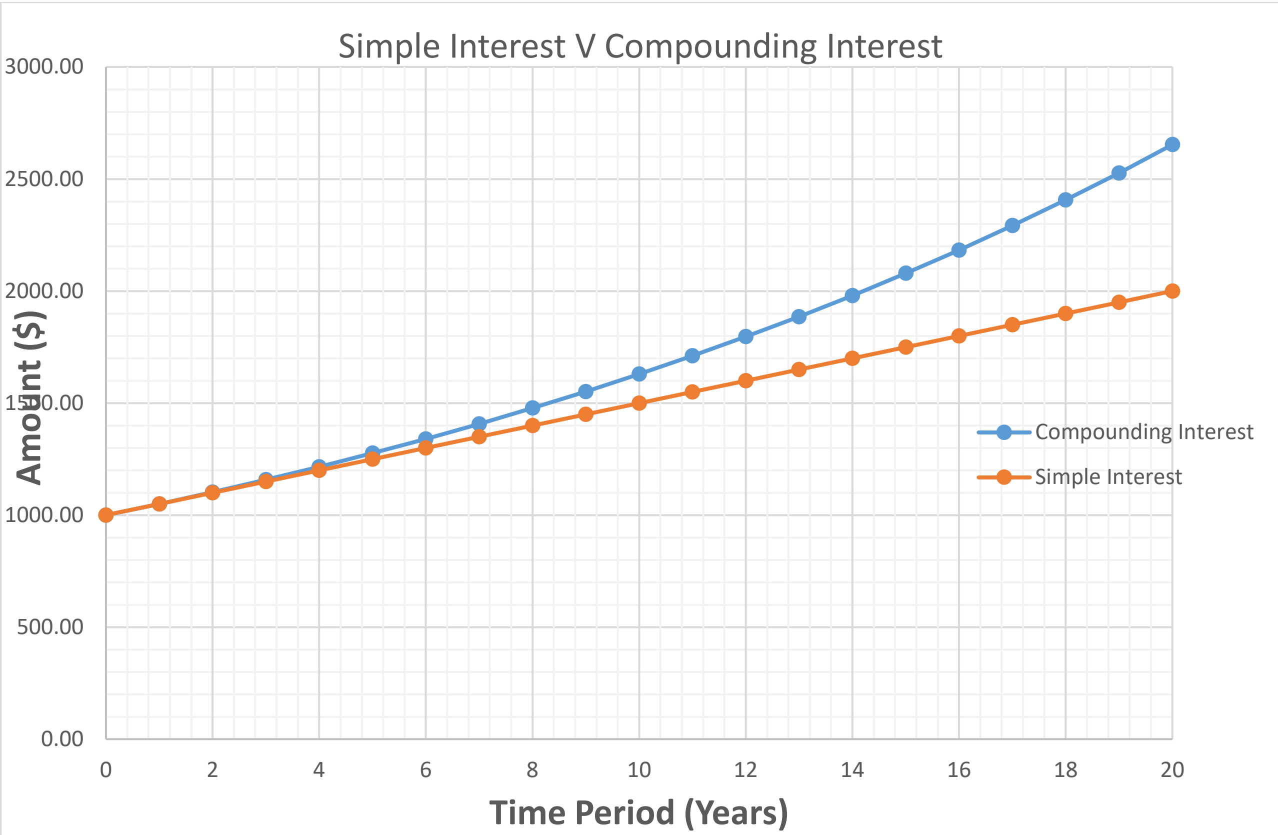
Graphical comparison between simple interest and compounding interest

The below graph shows the difference in return for:

- \$1000 at 5% p.a. simple interest
- \$1000 at 5% p.a. compounded annually

As can be seen the simple interest is a common difference recursion model with the following relation: $V_{n+1} = V_n + 50$, where $V_0 = 1000$

Whereas the compounding interest is a common ratio recursion model with the the following relation: $V_{n+1} = V_n(1.05)^n$, where $V_0 = 1000$



Example.2

A sum of \$4000 is invested for 10 years at a rate of 8.0% p.a., interest compounded annually.

Task.1 Generate the compound interest formula for this investment.

Task.2 Find the amount in the balance after 10 years and the interest earned over this period.

Task.1 Generate the compound interest formula for this investment.

$V_n = V_0(1 + \frac{r}{100})^n$

n = 10 years

r = 8.0% p.a.

V_0 = \$4000

$\therefore V_n = 4000(1 + \frac{8}{100})^n$

NB: This is the "general" form as it can be used for any value of n

Task.2 Find the amount in the balance after 10 years and the interest earned over this period.

$V_{10} = 4000(1.08)^{10}$
$= \underline{\$8635.70}$

I = $V_{10} - V_0$
= 8635.70 – 4000
= <u>\$4635.70</u>

At the completion of the 10 year investment the balance of the account would be \$8635.70 with \$4635.70 interest being earned.

Example.3

A sum of $6000 is invested for 12 years at 6% p.a., interest compounded quarterly. What is the balance of the account after the 12 year period?

NB: This questions is based upon **quarterly** calculations. So terms and rates must be **per quarter**.

$n = 12\ years$

$= 12 \times 4 = 48\ quarters$

$r = 8.0\%\ p.a.$

$= 8/4 = 2.0\%\ per\ quarter$

$V_0 = \$6000$

$V_n = V_0(1 + \frac{r}{100})^n$

$\therefore V_{48} = 6000(1 + \frac{2}{100})^{48}$

$= \$15522.42$

NB: Compounding interest calculations require you to change both the number of periods and rate to reflect the compounding period.

Eg. For compounded periods:

- quarterly: n = 4 x no. of years
- monthly: n = 12 x no. of years
- fortnightly n = 26 x no. of years
- weekly n = 52 x no. of years
- daily n = 365 x no. of years

Eg. For compounding rates

- quarterly: r = r (p.a.)/4
- monthly: r = r(p.a.)/12
- fortnightly r = r(p.a.)/26
- weekly r = r(p.a.)/52
- daily r = r(p.a)/365

Example.4

Find the principal that will grow to $5000 in 5 years, if interest is added monthly at 6% p.a.

NB: This questions is based upon **monthly** calculations. So terms and rates must be **per month**.

n = 5 years

= 5 x 12 = <u>60 months</u>

r = 6.0% p.a.

= 6/12 = <u>0.5% per month</u>

V_{60} = $5000

V_0 = ?

$V_n = V_0(1 + \frac{r}{100})^n$

$\therefore 5000 = V_0(1 + \frac{0.5}{100})^{60}$

$\therefore V_0 = \frac{5000}{(1+\frac{0.5}{100})^{60}}$

= $<u>3706.86</u>

Or alternatively:

$$\text{solve}\left(5000 = v \cdot \left(1 + \frac{0.5}{100}\right)^{60}, v\right) \qquad v = 3706.86$$

Application.2: Reducing balance depreciation
$U_{n+1} = RU_n$, where R < 1

Reducing balance depreciation, also known as ***diminishing value depreciation***, is where an item depreciates by a percentage of the previous book value.

Reducing balance depreciation can be modelled using a recursion rule:

$$\boxed{V_{n+1} = RV_n} \quad \text{or} \quad \boxed{V_{n+1} = (1 - \frac{r}{100})V_n}$$

Where V_n is the value of the assest after n depreciation periods, and;
$R = 1 - \frac{r}{100}$, where r is the depreciation rate.

The following equation can be used to calculate the future value (book value) of a depreciating item:

$$\boxed{V_n = V_o R^n} \quad \text{or} \quad \boxed{V_n = V_0(1 - \frac{r}{100})^n}$$

Where   is the value of the asset (book value)
n is the time since purchase
R is the rate of depreciation (= 1 - $\frac{r}{100}$)
V_0 is the cost price

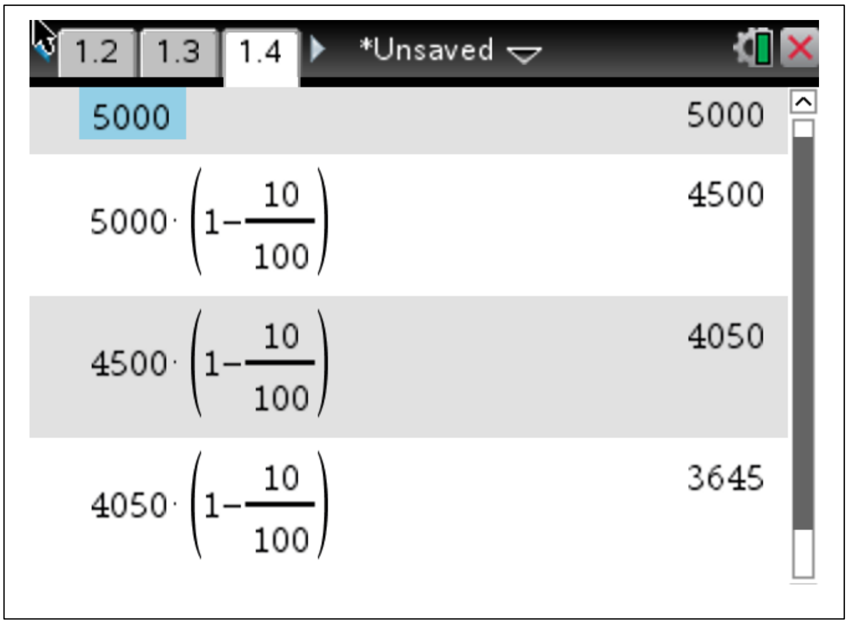
Example.5
Let's repeat a previous example where James purchases a new car valued at $50 000. Only this time for taxation purposes James chooses to depreciate his car using the ***reducing balance method***. The depreciation was 10% of the previous book value.

1. Draw a depreciation schedule for the first 5 years of the car
2. What is the book value after 5 years
3. Construct a graph of book value against time

Part.1

$d_1 = 10\%$ of 50000
$= \$5000$

$\therefore V_1 = 50000 - 5000$
$= \$45000$



$d_2 = 10\%$ of 45000
$= \$4500$

$\therefore V_2 = 45000 - 4500$
$= \$40500$

$d_3 = 10\%$ of 40500
$= \$4050$

$\therefore V_3 = 40500 – 4050$
$= \$36450$

$d_4 = 10\%$ of 36450
$= \$3645$

$\therefore V_4 = 36450 - 3645$
$= \$32805$

$d_5 = 10\%$ of 32805
$= \$3280.50$

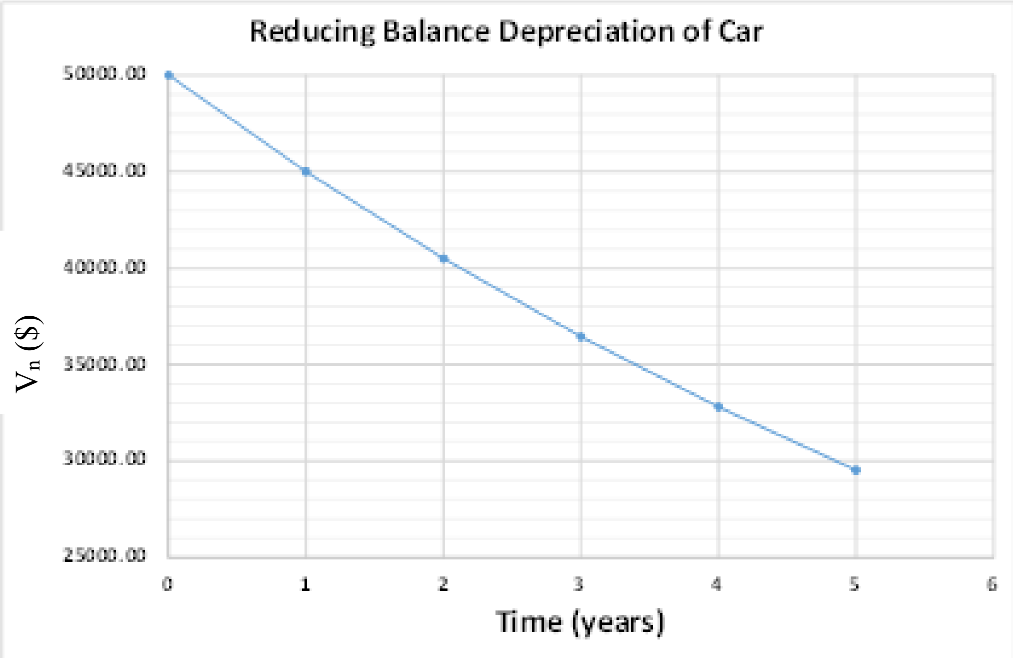
$\therefore V_5 = 32805 - 3280.50$
$= \$29524.$

Part.2

The book value of the car after 5 years is $29524.50

Part.3

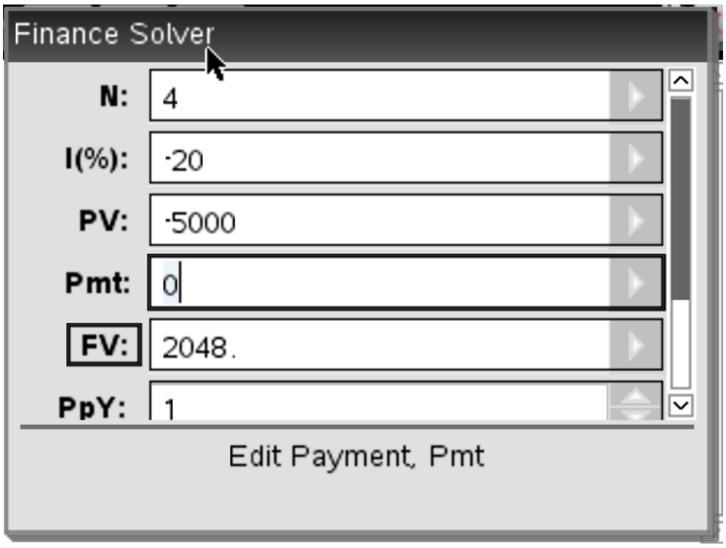
Graph showing future value (V_n) against time



Example.6

Let's repeat a previous example where Naomi originally purchased her car for \$5000. Given it is now depreciating via the ***reducing balance method*** at 20% p.a., what will be the book value and total depreciation of the car after 4 years?

Part.1



V_4 = ?
P = \$5000
r = 20% p.a.
n = 4 years

$$V_n = P(1 - \frac{r}{100})^n$$
$$V_4 = 5000(1 - \frac{20}{100})^4$$
$$= \$2048$$

The car's book value after 4 years

Part.2

Total depreciation = V_0 - V_4
= 5000 - 2048
= \$2952

The total depreciation after 4 years will be \$2952

Exam Styled Questions – Multiple Choice

Use the following information to answer Questions 1 and 2.

Shirley would like to purchase a new home. She will establish a loan for $225 000 with interest charged at the rate of 3.6% per annum, compounding monthly.

Each month, Shirley will pay only the interest charged for that month.

Question 1
(2017 VCAA Exam 1 Section A - Qn 19)

After three years, the amount that Shirley will owe is

A. $73 362
B. $170 752
C. $225 000
D. $239 605
E. $245 865

$\boxed{C}$

Shirley is only paying back the interest charged, she will never reduce the actual loan amount (principal)

Question 2
(2017 VCAA Exam 1 Section A - Qn 20)

Let V_n be the value of Shirley's loan, in dollars, after n months.
A recurrence relation that models the value of V_n is

A. $V_0 = 225\,000, \ V_{n+1} = 1.003\, V_n$
B. $V_0 = 225\,000, \ V_{n+1} = 1.036\, V_n$
C. $V_0 = 225\,000, V_{n+1} = 1.003\, V_n - 8100$
D. $V_0 = 225\,000, \ V_{n+1} = 1.003\, V_n - 675$
E. $V_0 = 225\,000, \ V_{n+1} = 1.036\, V_n - 675$

$\boxed{D}$

The interest is compounded monthly, so the interest rate of 3.6% p.a. must be first converted to a rate per month.

$3.6\% \ (p.a.) = \frac{3.6}{12} = 0.3\% \ p.month$

Payment per month = 0.3% of $225,000 $\frac{0.3}{100} \times 225000 = \$675.00$

∴ recursion relation is Option D. The value of the loan is being increased by a growth factor of 1.003 per month, but also reduced by a payment of $675.00 (0.3% of $225,000).

Question 3
(2016 VCAA Exam 1 Section A - Qn 19)

The purchase price of a car was \$26 000.
Using the reducing balance method, the value of the car is depreciated by 8% each year.
A recurrence relation that can be used to determine the value of the car after n years, C_n , is

A. $C_0 = 26\,000,\ C_{n+1} = 0.92\,C_n$
B. $C_0 = 26\,000, C_{n+1} = 1.08\,C_n$
C. $C_0 = 26\,000, C_{n+1} = C_n + 8$
D. $C_0 = 26\,000, C_{n+1} = C_n - 8$
E. $C_0 = 26\,000, C_{n+1} = 0.92\,C_n - 8$

A

A depreciation of 8% each year, represents a growth factor of $(1 - \frac{8}{100})$ *= 0.92. ∴ Option A.*

Question 4
(2016 VCAA Sample Exam Section A - Qn 17)

$$P_0 = 2000, P_{n+1} = 1.5P_n - 500$$

The first three terms of a sequence generated by the recurrence relation above are

A. 500, 2500, 2000 …
B. 2000, 1500, 1000 …
C. 2000, 2500, 3000 …
D. 2000, 2500, 3250 …
E. 2000, 3000, 4500 …

2000	2000
2000· 1.5−500	2500.
2500.· 1.5−500	3250.

D

Question 5
(2016 VCAA Sample Exam Section A - Qn 18)

Which of the following recurrence relations will generate a sequence whose values decay geometrically?

A. $L_0 = 2000, L_{n+1} = L_n - 100$
B. $L_0 = 2000, L_{n+1} = L_n + 100$
C. $L_0 = 2000, L_{n+1} = 0.65L_n$
D. $L_0 = 2000, L_{n+1} = 6.5L_n$
E. $L_0 = 2000,\ L_{n+1} = 0.85L_n - 100$

C

Option C is the only geometric sequence that contains an R value < 1 (R = 0.65) this will produce a decaying sequence.

Question 6
(2016 VCAA Sample Exam Section A - Qn 20)

Rohan invests \$15 000 at an annual interest rate of 9.6% compounding monthly.
Let V_n be the value of the investment after n months. A recurrence relation that can be used to model this investment is

A. $V_0 = 15\ 000,\ V_{n+1} = 0.96V_n$
B. $V_0 = 15\ 000,\ V_{n+1} = 1.008V_n$
C. $V_0 = 15\ 000,\ V_{n+1} = 1.08V_n$
D. $V_0 = 15\ 000,\ V_{n+1} = 1.0096V_n$
E. $V_0 = 15\ 000,\ V_{n+1} = 1.096V_n$

$\boxed{B}$

The interest is compounded monthly, so the annual interest rate of 9.6% p.a. must be first converted to a rate per month.

$9.6\%\ (p.a.) = \frac{9.6}{12} = 0.8\%\ p.month$

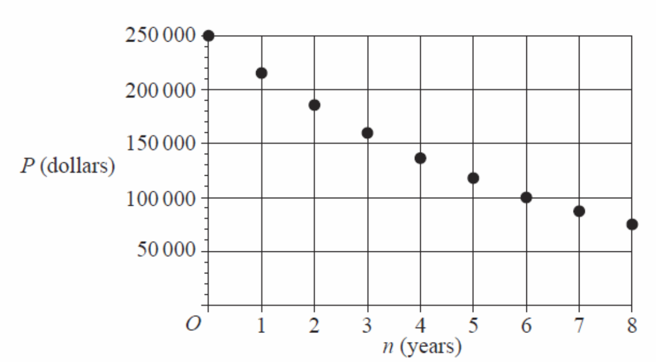
The growth factor (R) = $1 + \frac{0.8}{100} = 1.008$

R > 1 as it is a growing investment.

∴ recursion relation is Option B.

Question 7
(2016 VCAA Sample Exam Section A - Qn 24)

The following graph shows the decreasing value of an asset over eight years.



Let P_n be the value of the asset after n years, in dollars.
A rule for evaluating P_n could be

A. $P_n = 250\ 000 \times (1 + 0.14)^n$
B. $P_n = 250\ 000 \times 1.14 \times n$
C. $P_n = 250\ 000 \times (1 - 0.14) \times n$
D. $P_n = 250\ 000 \times (0.14)^n$
E. $P_n = 250\ 000 \times (1 - 0.14)^n$

$\boxed{E}$

This is a non linear geometric sequence decay.
∴ option E