Section 3.3.11 – Einstein's Special Relativity (Mass-Energy)

Mass – Energy for stationary objects

$$\boldsymbol{E_0 = mc^2}$$

Where E_0 represents rest energy (Joules)
m represents mass (kilograms)
c represents the speed of light (3.0×10^8 ms^{-1})

NB: Rest energy is the total energy of a stationary mass.
Rest energy can also be represented by E_{rest}

This famous equation explains the relationship between mass and energy. In essence, mass and energy are one of the same, or if you like two different "versions" of the same thing. Or, if you like the "equivalent" energy of a given mass.

Einstein's connection between mass and energy is similar to Maxwell's connection between the seemingly independent concepts of electric and magnetic in his electromagnetic theory of radiation.

Einstein explained that "*mass is a property of the way that energy behaves under certain conditions*".

NB: When you square c you get an extremely large value. ie. $(3.0 \times 10^8)^2 = 9.0 \times 10^{16}$
$\therefore$ even the smallest amount of mass will convert to an extremely large quantity of energy.

Example.1
Calculate the rest energy (E_0) of an electron of mass (m_e = 9.1×10^{-31} kg)

$E_0 = ?$
$m_e = 9.1 \times 10^{-31}\ kgE$
$c = 3.0 \times 10^8\ ms^{-1}$

$$\begin{aligned} E_0 &= mc^2 \\ &= 9.1 \times 10^{-31} \times (3.0 \times 10^8)^2 \\ &= 9.1 \times 10^{-31} \times (9.0 \times 10^{16}) \\ &= 8.19 \times 10^{-14}\ \text{J} \end{aligned}$$

Example.2
How much rest mass is equivalent to 5.0 Joules of energy?

$m = ?$
$E_0 = 5.0\,J$
$c = 3.0 \times 10^8\ ms^{-1}$

$$E_0 = mc^2$$

$$\therefore m = \frac{E_0}{c^2}$$

$$= \frac{5}{(3.0 \times 10^8)^2}$$

$$= 5.6 \times 10^{-17} kg$$

Mass – Energy for moving objects

When an object is moving, its kinetic energy (E_k) is added to its rest energy (E_0 or E_{rest}) in order to calculate its total energy (E_{total}).
The total energy of an object can be calculated using the following equation:

$$\boldsymbol{E_{total} = E_k + E_0 = \gamma mc^2}$$

Where E_{total} represents the objects total energy (Joules)
E_k represents the objects kinetic energy (Joules)
E_0 represents the rest energy of an object (Joules)
γ represents the Lorentz factor
m represents the mass of the object (kg)
c represents the speed of light ($3.0 \times 10^8\ ms^{-1}$)

Recall

$$\boldsymbol{\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

As an object approaches the speed of light, its ***Lorentz factor (γ)*** will increase and so too will the object's total energy (E_{total}). In fact, if the object's speed were to approach the speed of light (c), it's energy would approach an infinite value.

It would take an ***infinite amount of energy*** to accelerate an object or particle to the ***speed of light*** (c). Accordingly, the speed of light is the universal speed limit.

Example.3

A linear accelerator (LINAC) can accelerate electrons to a speed of $0.98\,c$. Calculate the total energy of the electron travelling at this speed.
($m_e = 9.1 \times 10^{-31} kg$)

$E_{total} = ?$
$v = 0.98\,c$
$m = 9.1 \times 10^{-31} kg$

NB: $c^2 = (3.0 \times 10^8)^2 = 9.0 \times 10^{16}$

$$E_{total} = \gamma mc^2$$

$$E_{total} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \times mc^2$$

$$= \frac{1}{\sqrt{1 - \frac{(0.98c)^2}{c^2}}} \times (9.1 \times 10^{-31} \times 9.0 \times 10^{16})$$

$$= \frac{1}{\sqrt{1 - \frac{0.9604c^2}{c^2}}} \times (8.19 \times 10^{-14})$$

$$= \frac{1}{\sqrt{1 - \frac{0.9604\cancel{c^2}}{\cancel{c^2}}}} \times (8.19 \times 10^{-14})$$

$$= \frac{1}{\sqrt{1 - 0.9604}} \times (8.19 \times 10^{-14})$$

$$= \frac{1}{\sqrt{0.0396}} \times (8.19 \times 10^{-14})$$

$$= 5.025 \times (8.19 \times 10^{-14})$$

$$= 4.12 \times 10^{-13}\ J$$

Example.4

An electron has a kinetic energy of $5.0 \times 10^{-15}\,J$. Calculate the total energy of the electron. $(m_e = 9.1 \times 10^{-31} kg)$

$E_{total} = ?$
$E_k = 5.0 \times 10^{-11}\,J$
$m = 9.1 \times 10^{-31} kg$
$c = 3.0 \times 10^{8} ms^{-1}$
$\therefore c^2 = 9.0 \times 10^{16}$

$E_{total} = mc^2 + E_k$
$E_{total} = (9.1 \times 10^{-31} \times 9.0 \times 10^{16}) + 5.0 \times 10^{-15}$
$E_{total} = 8.19 \times 10^{-14} + 5.0 \times 10^{-15}$
$E_{total} = 8.69 \times 10^{-14}\,J$

Kinetic Energy

Traditional Newtonian physics, where objects are not travelling as speed near or approaching the speed of light, uses the following equation for kinetic energy:

$$\boldsymbol{E_k = \frac{1}{2}mv^2}$$

However, when objects are travelling at extremely fast (***relativistic***) speeds, the following equation must be used:

$$\boldsymbol{E_{total} = E_k + E_0 = \gamma mc^2}$$

Let's now establish a new equation for kinetic energy (E_k) using the previous equation.

$E_k + E_0 = \gamma mc^2$
$E_k + mc^2 = \gamma mc^2$
$E_k = \gamma mc^2 - mc^2$
$E_k = (\gamma - 1)mc^2$

$$\boldsymbol{E_k = (\gamma - 1)mc^2}$$

Where E_k represents the kinetic energy at high speeds (Joules)
γ represents the Lorentz Factor
m represents the rest mass (kg)
c represents the speed of light ($3.0 \times 10^8\ ms^{-1}$)

Example.5

Scientists accelerate an electron to relativistic speed of $2.5 \times 10^8\ ms^{-1}$. What is the kinetic energy of the electron at this speed?
($m_e = 9.1 \times 10^{-31} kg$)

$E_k = ?$
$v = 2.5 \times 10^8\ ms^{-1}$
$m_e = 9.1 \times 10^{-31} kg$
$c = 3.0 \times 10^8\ ms^{-1}$

Step 1 Find γ

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{(2.5 \times 10^8)^2}{(3.0 \times 10^8)^2}}}$$

$$\gamma = 1.81$$

Step 2 Find E_k

$$E_k = (\gamma - 1)mc^2$$

$$= (1.81 - 1) \times 9.1 \times 10^{-31} \times 9.0 \times 10^{16}$$

$$= 6.63 \times 10^{-14}\ J$$

Exam Styled Questions

Question 1 (VCAA 2018 Physics Exam Qn 14 MC)
Which one of the following statements about the kinetic energy, E_k, of a proton travelling at relativistic speed is the most accurate?

A. The difference between the proton's relativistic E_k and its classical E_k cannot be determined.
B. The proton's relativistic E_k is greater than its classical E_k.
C. The proton's relativistic E_k is the same as its classical E_k.
D. The proton's relativistic E_k is less than its classical E_k.

B

At $v << c$ the relativistic E_k is the same as the classical E_k.
However, as v approaches c the relativistic E_k increases at a greater rate than classical E_k.

Question 2 (VCAA 2018 Physics Exam Qn 15 SA)
A stationary scientist in an inertial frame of reference observes a spaceship moving past her at a constant velocity. She notes that the clocks on the spaceship, which are operating normally, run eight times slower than her clocks, which are also operating normally. The spaceship has a mass of 10 000 kg. Calculate the kinetic energy of the spaceship in the scientist's frame of reference. Show your working.

$\gamma = 8$, from the phrase '…runs eight times slower…'
$E = (\gamma - 1)mc^2$
$E = (8 - 1)(10\ 000)(3 \times 10^8)^2$
$E = 6.3 \times 10^{21}\ J$

$\mathbf{6.3 \times 10^{21}}$ **J**

Question 3 (VCAA 2019 Physics NHT Qn 18 MC)
If a particle's kinetic energy is 10 times its rest energy, E_{rest} , then the Lorentz factor, γ, would be closest to

A. 9
B. 10
C. 11
D. 12

$E = (\gamma - 1)mc^2$
$\Rightarrow (\gamma - 1) = 10$
$\therefore \gamma = 11$

11

Question 4 (VCAA 2018 Physics NHT Qn 15 SA)

An unstable subatomic particle, known as a π_0 meson, decays completely into electromagnetic radiation. The mass of this π_0 meson is 2.5×10^{-28} kg. How much energy would be released by this π_0 meson if it decays at rest?

$$E_0 = mc^2$$
$$E_0 = 2.5 \times 10^{-28} \times 9 \times 10^{16}$$
$$E_0 = 2.3 \times 10^{-11} J$$

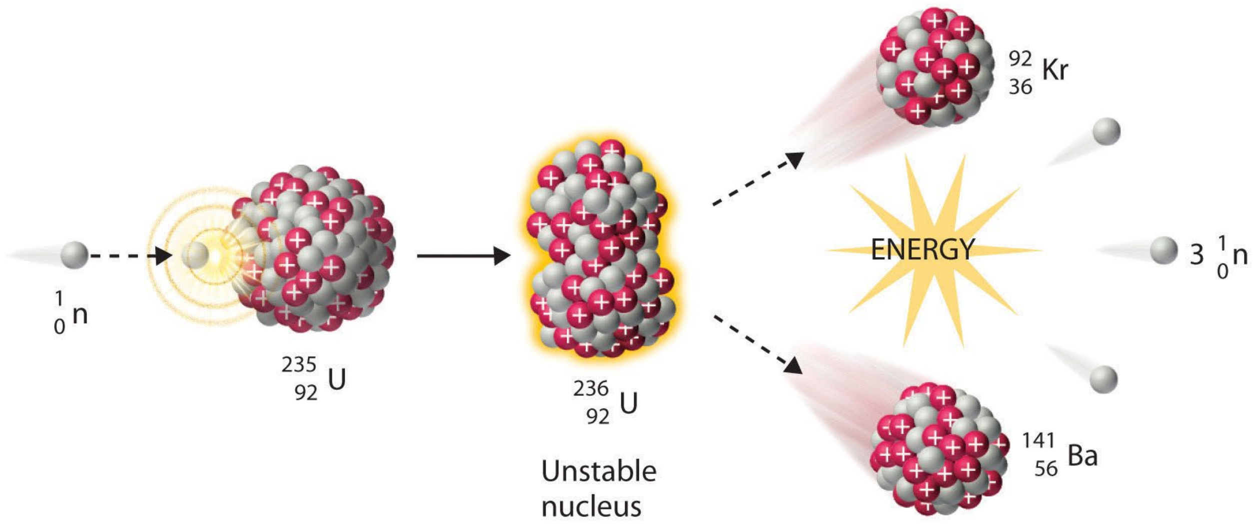
$$\mathbf{2.3 \times 10^{-11}\ J}$$

Nuclear Reactions

$E_0 = mc^2$ is an extremely useful equation that can be used to calculate the amount of energy produced in a nuclear reaction.

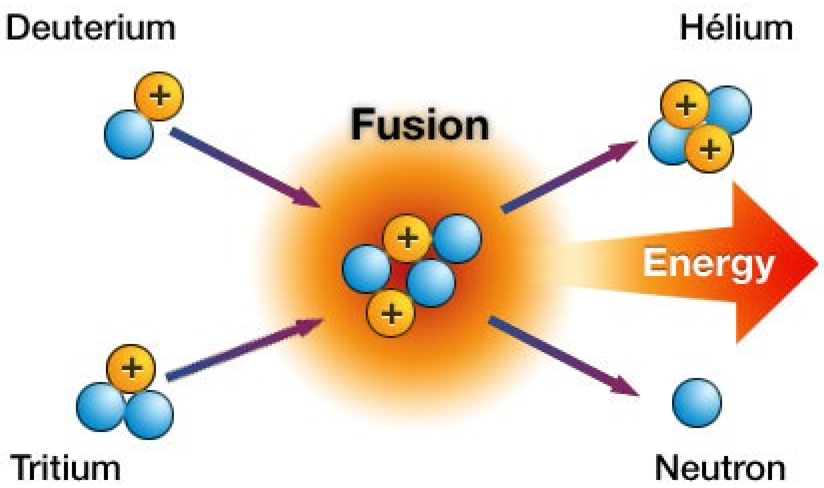
Nuclear reactions are classified as either:

- ***Nuclear Fission***, where a heavy nuclei is ***split*** into multiple lighter nuclei
  Eg. the fission of U-235, via neutron bombardment



$$ {}_0^1n + {}_{92}^{235}U \rightarrow {}_{36}^{92}Kr + {}_{56}^{141}Ba + 3{}_0^1n + Energy $$

- ***Nuclear Fusion***, where two nuclei ***come together*** to form a heavier nuclei
  Eg. the fusion of Deuterium (H-2) and Tritium (H-3), 2 isotopes of Hydrogen



$$ {}_1^2H + {}_1^3H \rightarrow {}_2^4He + {}_0^1n + Energy $$

Energy conversion in nuclear reactions

Regardless of the classification of nuclear reaction both experience a *change in mass (Δm)* as a result. The sum of the mass of nuclei before the reaction is always *slightly greater* than the sum of the mass of the nuclei after the reaction.

This extremely small *mass loss* is converted in *energy* via the equation $E_0 = mc^2$.
That is the energy released has been formed by the mass lost.

NB: It only takes an extremely small mass loss to generate considerable energy due to the multiplication of c^2 (9.0×10^{16}).

Example.1

Radium-226 undergoes nuclear fission via the below nuclear equation:

$$^{226}_{88}Ra \rightarrow \, ^{222}_{86}Rn + \, ^{4}_{2}He + Energy$$

Given its mass loss in the process is $8.61818 \times 10^{-30}\ kg$, calculate the energy released in the nuclear reaction.

$m = 8.61818 \times 10^{-30}\ kg$
$c = 3.0 \times 10^{8}\ ms^{-1}$
$E_0 = ?$

$E_0 = mc^2$
$= 8.61818 \times 10^{-30} \times (3.0 \times 10^{8})^2$
$= \mathbf{7.76 \times 10^{-13} J}$

Example.2

Calculate the energy released when Deuterium is fused with Tritium via the below equation. The total mass of reactant nuclei before the reaction is $8.350 \times 10^{-27}\ kg$ and the total mass of product nuclei after the reaction is $8.320 \times 10^{-27}\ kg$.

$$^{2}_{1}H + \, ^{3}_{1}H \rightarrow \, ^{4}_{2}He + \, ^{1}_{0}n + Energy$$

$E_0 = ?$
$m_{reactants} = 8.350 \times 10^{-27} kg$
$m_{products} = 8.320 \times 10^{-27}\ kg$
$c = 3.0 \times 10^{8}\ ms^{-1}$

Step.1 Find Δm

$\Delta m = m_{reactants} - m_{products}$
$= 8.350 \times 10^{-27} - 8.320 \times 10^{-27}$
$= 0.03 \times 10^{-27}$ kg

Step.2 Find E_0

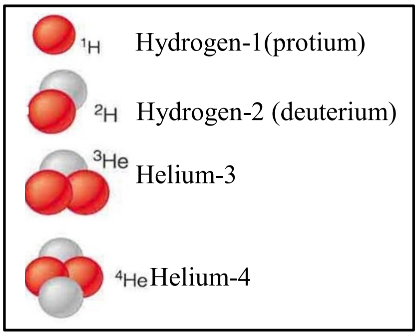
$E_0 = mc^2$
$= 0.03 \times 10^{-27} \times (3.0 \times 10^{8})^2$
$= \mathbf{2.7 \times 10^{-12}\ J}$

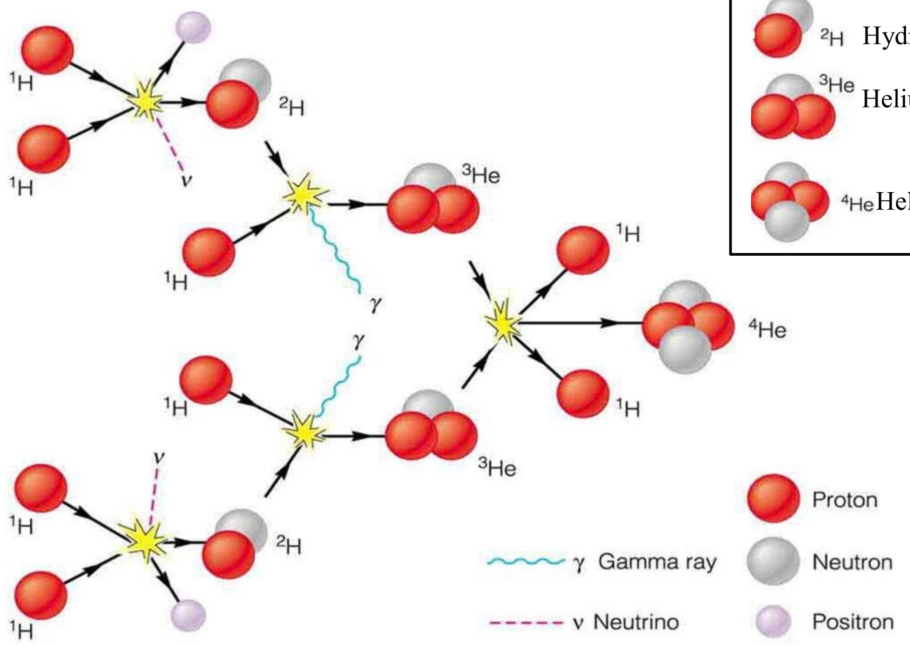
Nuclear fusion in the sun

Our Sun is powered by a series of ***nuclear fusion*** reactions. The Sun continuously converts mass-energy stored as mass into radiant light and heat in the form of ***electromagnetic radiation***.

The Sun consists mostly of hydrogen plasma which undergoes nuclear fusion to form the slightly heavier element Helium. Helium nuclei also undergoes nuclear fusion to generate even heavier nuclei and the cycle continues to generate a range of heavier nuclei.

With each individual nuclear fusion there is mass lost. It has been calculated that the Sun loses $4.4\,Tg\ (4.4\ \times 10^{9}\ kg)$ per second. Even at this incredible rate the Sun. with a mass of $2.0\times 10^{30}\ kg$, can exist for a further 4.5 billion years.





The formation of heavier nuclei via the process of fusion between protons

Example.3

An electron and its antiparticle, a positron, each at rest, annihilate to produce two photons (gamma ray) whose total energy is 1.6352×10^{-13} J. Apart from the protons, nothing else is produced in this process. The mass of the electron and positron are the same.

Calculate the mass of the electron.

$m_e = ?$

$E_\gamma = 1.6352\ \times 10^{-13}\,J$

$c = 3.0 \times 10^{8}\ ms^{-1}$

Step.1 Find the m_{total}

$E_0 = mc^2$

$\therefore\ m_{total} = \dfrac{E}{c^2}$

$m_{total} = \dfrac{1.6352\ \times 10^{-13}}{(3.0\ \times 10^{8})^2}$

$= 1.82 \times 10^{-30}\ kg$

Step.2 Find m_e

$m_e = \dfrac{m_{total}}{2}$

$= \dfrac{1.82\ \times 10^{-30}}{2}$

$= \mathbf{9.1 \times 10^{-31} kg}$

Exam Styled Questions

Question 1 (VCAA 2017 Physics Exam Qn 11 MC)
On average, the sun emits 3.8×10^{26} J of energy each second in the form of electromagnetic radiation, which originates from the nuclear fusion reactions taking place in the sun's core. The corresponding loss in the sun's mass each second would be closest to

A. 2.1×10^9 kg
B. 4.2×10^9 kg
C. 8.4×10^9 kg
D. 2.1×10^{12} kg

$E = mc^2$

$3.8 \times 10^{26} = m \times (3 \times 10^8)^2$

$m = 4.2 \times 10^9\ kg$

B

Question 2 (VCAA 2019 Physics NHT Qn 19 SA)
In a nuclear fusion reaction in the sun's core, two deuterium nuclei, each with a mass of 3.3436×10^{-27} kg, fuse to produce one helium-4 nucleus with a mass of 6.6465×10^{-27} kg. Ignore the kinetic energy of the nuclei before the reaction. Calculate the energy released. Show your working

Mass change = $(2 \times 3.3436 \times 10^{-27}) - 6.6465 \times 10^{-27}$
Mass change = $4.07 \times 10^{-29}\ kg$

$E_0 = mc^2$
$E_0 = 4.07 \times 10^{-29} \times 9 \times 10^{16}$
$E_0 = 3.7 \times 10^{-12} J$

$\mathbf{3.7 \times 10^{-12}}$ **J**

Question 3

A helium ion is accelerated from a speed of 9.0×10^{7} ms^{-1} to a speed of 1.5×10^{8} ms^{-1}. Scientists calculate accurately the work done on the helium ion during this acceleration.

Data
- Mass of a helium ion: 6.64×10^{-27} kg
- Rest mass energy of a helium ion: 5.98×10^{-10} J
- At a speed of 9.0×10^{7} ms^{-1}, $\gamma = 1.05$; at a speed of 1.5×10^{8} ms^{-1}, $\gamma = 1.15$

Which of the options below is the best estimate of the answer they obtain?

A. 2×10^{-19} J
B. 6×10^{-11} J
C. 1×10^{-9} J
D. 6×10^{-9} J

Work done = change in KE, KE = $(\gamma - 1)\, mc^2$, so
Work done = $((1.15 - 1) - (1.05 - 1)) \times mc^2$, which gives $0.1 \times m_o c^2 = 6 \times 10^{-11}$ J.

B

Question 4

Muons and antimuons are anti-particles of each other. They have the same mass. When a muon meets an antimuon, both are destroyed and two photons (gamma rays) are formed. If the two particles are effectively stationary, then the two photons have a **total** energy of 3.38×10^{-11} J. Using this data, which one of the following is closest to the mass of a single muon?

A. 3.76×10^{-28} kg
B. 1.88×10^{-28} kg
C. 1.13×10^{-19} kg
D. 5.64×10^{-19} kg

Using $\Delta E = mc^2$, $3.38 \times 10^{-11} = 2 \times m \times 9 \times 10^{16}$, so $m = 1.88 \times 10^{-28}$ kg

B

Question 5

In a particle accelerator, an alpha particle of mass 6.64424×10^{-27} kg is accelerated from rest to high speed. The total work done on the alpha particle is equal to 7.714×10^{-10} J.
Which one of the following is closest to its final speed?

A. 0.90c
B. 0.95c
C. 0.85c
D. 0.80c

Using $KE = (\gamma - 1)mc^2$, $\gamma = [7.714 \times 10^{-10} / (6.64424 \times 10^{-27} \times 9 \times 10^{16})] + 1 = 2.29$, *which means* $v = 0.9c$.

A

Question 6

In a nuclear reactor, some mass is converted into energy. Which one of the following is the best approximation to the total energy released when 1 kg of mass is totally converted into energy?

A. 10^5 J
B. 10^9 J
C. 10^{13} J
D. 10^{17} J

Using $E_0 = mc^2$, $E_0 = 1.0 \times (3 \times 10^8)^2$

D

Question 7

Which one of the following is closest to the work that must be done on a proton to increase its speed from zero to 0.9 c, that is $\gamma = 2.29$? (Take $m_{proton} = 1.67 \times 10^{-27}$ kg.)

A. 1.9×10^{-10} J
B. 4.0×10^{-20} J
C. 3.5×10^{-11} J
D. 1.7×10^{-27} J

Work done = total energy – rest energy = $mc^2(\gamma\text{-}1)$

A