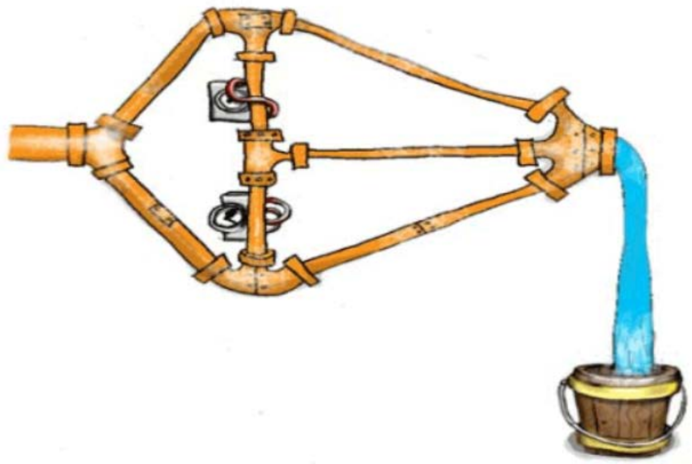
Section 4.2.4 Minimum Cut & Maximum Flow

VCAA "Dot Points"

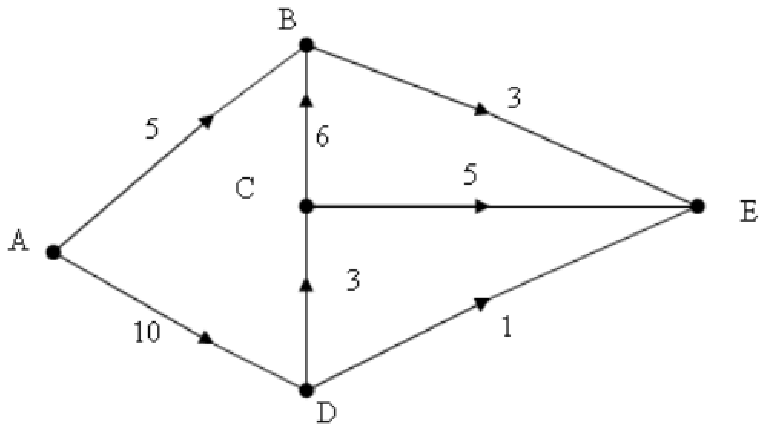
Networks and decision mathematics, including:
- *the flow problem, and the minimum cut/maximum flow theorem*
- *recognise that a problem is an example of the flow problem, use networks to model flow problems and determine the minimum flow problem by inspection, or by using the minimum cut/maximum flow theorem for larger scale problems*

Maximum Flow

In further mathematics, directed graphs can represent situations in which 'things' **flow** from a start point (called a **source**) to a finish point (called a **sink**), via 'channels'.

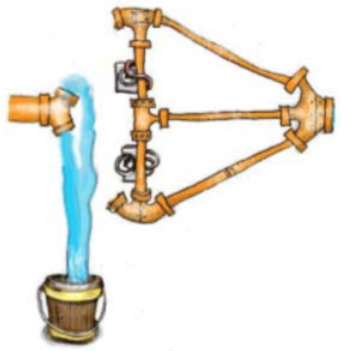


In the above example, water is flowing through the pipes into a bucket. This situation could be represented by a **directed graph** where the numbers indicate the **amount of water** that can travel through that channel (eg Litres per second).

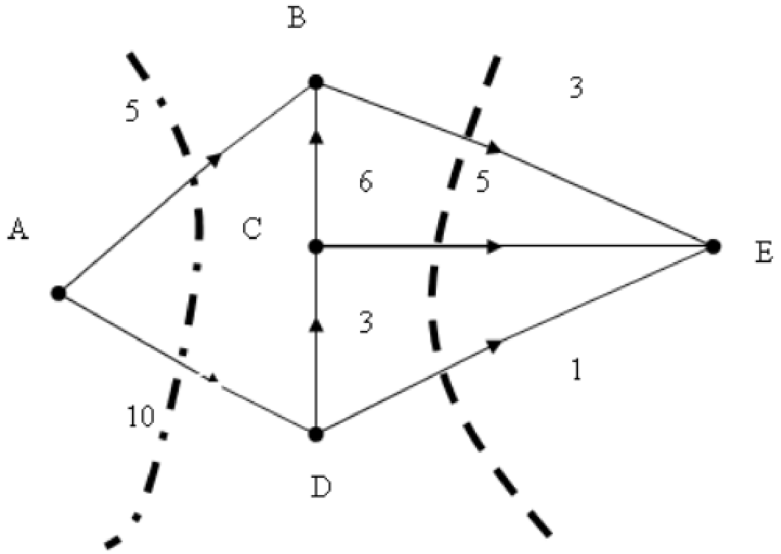


Let's now use the "**minimum cut**" technique to calculate the "**maximum flow**" rate of this pipe system.

To construct a cut correctly, you need to **draw a line** that completely "**cuts off**" the flow of water between A (**source**) and E (**sink**). It would be like knocking out the right pipes in real life:



In the above two situations the flow was completely cut off. The cuts would look like this on the graph:

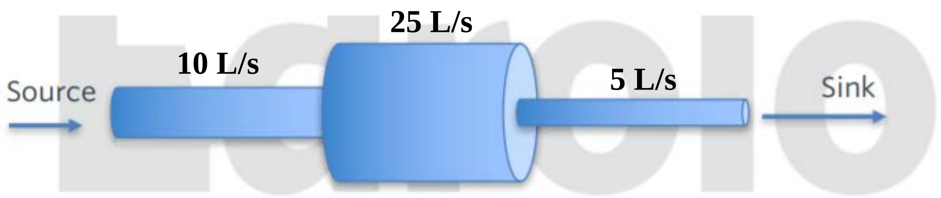


You now simply adds up the **weights** of the channels that have been cut:

The first cut has a capacity of (5 + 10) = 15
The second cut has a capacity of (3 + 5 + 1) = 9

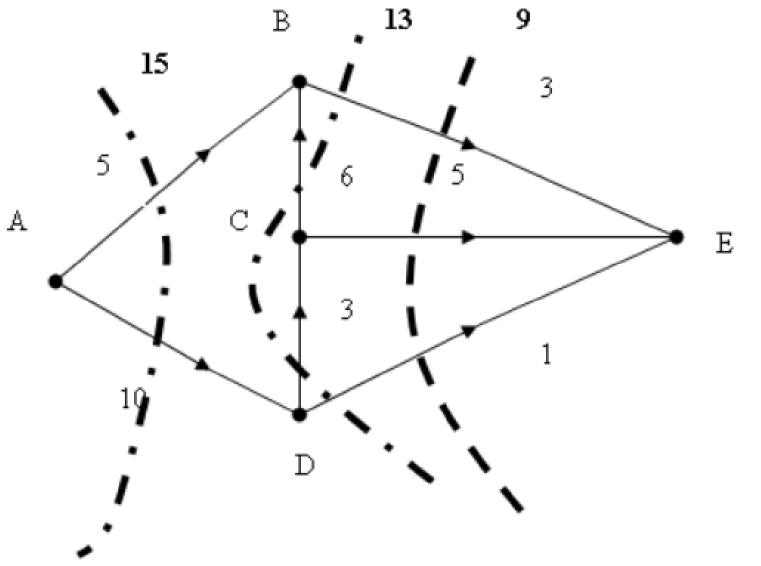
NB: The cut that has the **minimum capacity** simplifies the graph correctly. Just remember that it's the "small pipes that restrict the amount of water that flows to the bucket".

Example.1
What is the maximum flow rate of the below system?



The pipe with the minimum capacity determines the maximum flow rate.
Therefore, the maximum flow rate for the above system is 5 L/s.

Be sure to examine all possible cuts that completely "**cut off**" the flow.
Consider the following new cut.

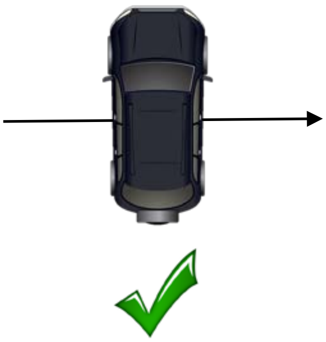


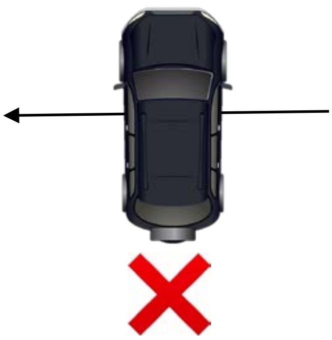
At first glance it would appear that the new cut has a capacity of (1 + 3 + 6 + 3) = 13.

But when you consider the network more closely you will notice that only three of the four edges actually impact upon the capacity. The edge with a weight of 6 shouldn't have counted because with edge below it cut, no water can reach it. The **capacity of the cut** would then be **7**.

So the **minimum cut** and therefore **maximum capacity** of this network is **7**.

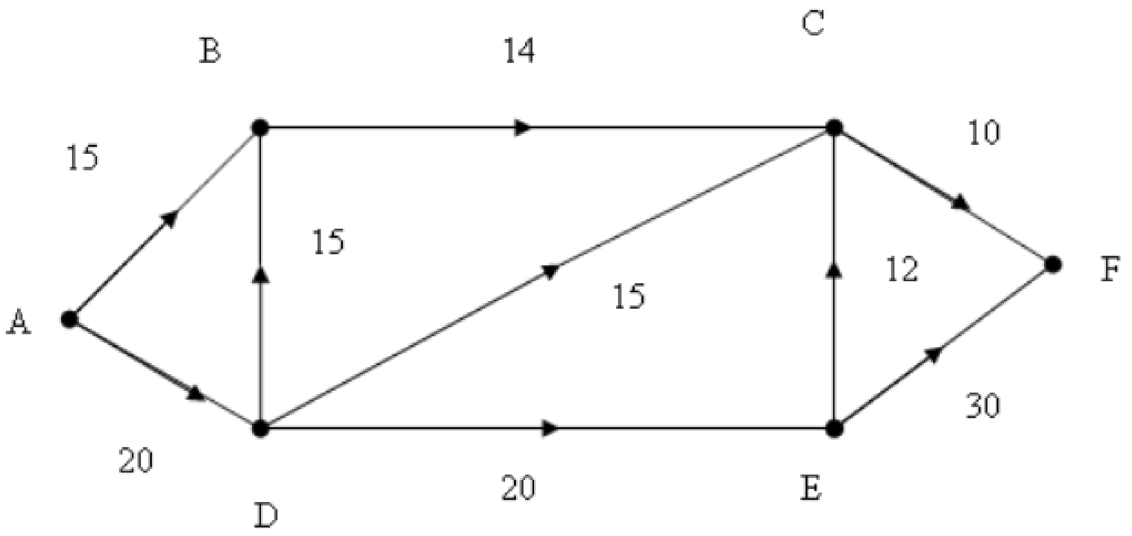
NB: Imagine driving a car along the a cut from the <u>bottom of the network to the top.</u>
Edges that are directed from your **left to right are counted**
Edges that are directed from your **right to left are not counted**



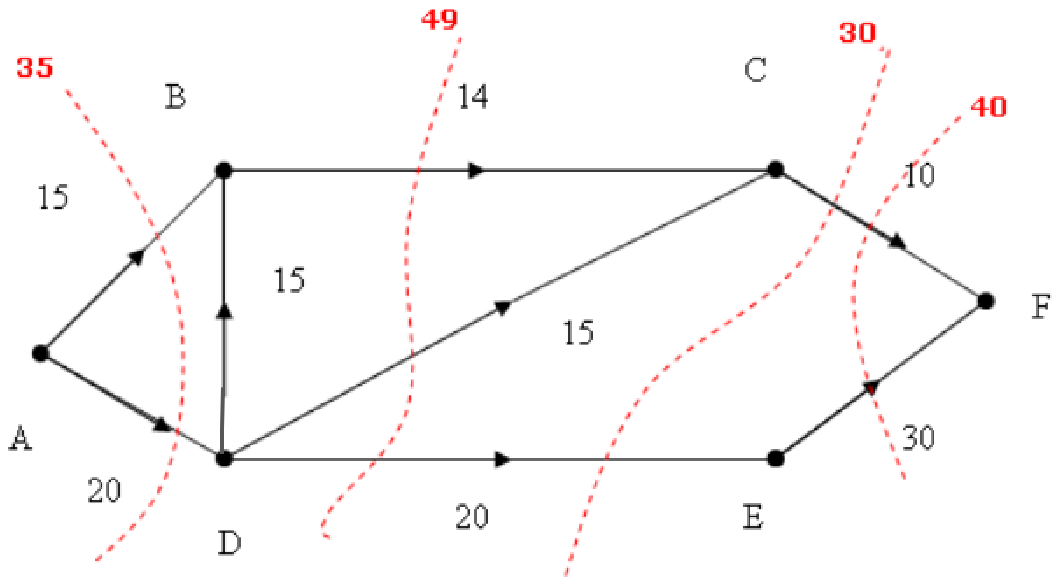


Example.2

What is the maximum flow from A to F for this network?



To find the maximum flow, first identify the minimum cut.

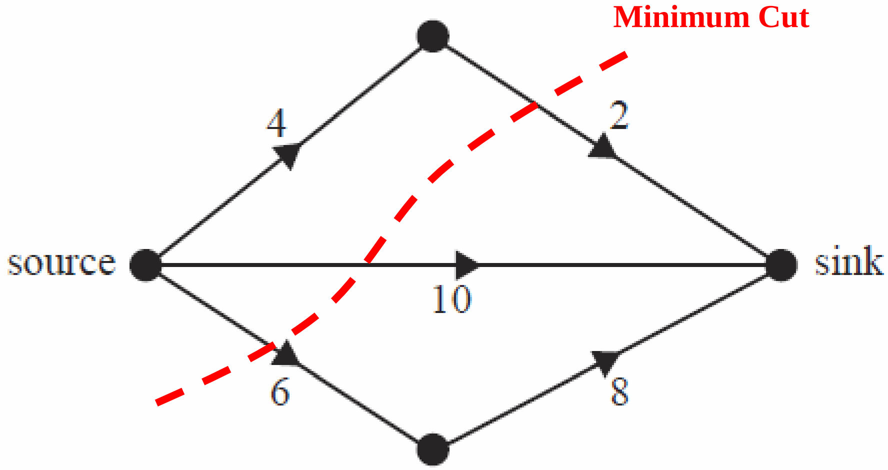


Therefore the minimum cut and maximum flow is 30.

Exam Styled Questions – Multiple Choice

Question 1
(2016 Exam 1, Module 2, Qn 2)

The following directed graph shows the flow of water, in litres per minute, in a system of pipes connecting the source to the sink.



The maximum flow, in litres per minute, from the source to the sink is

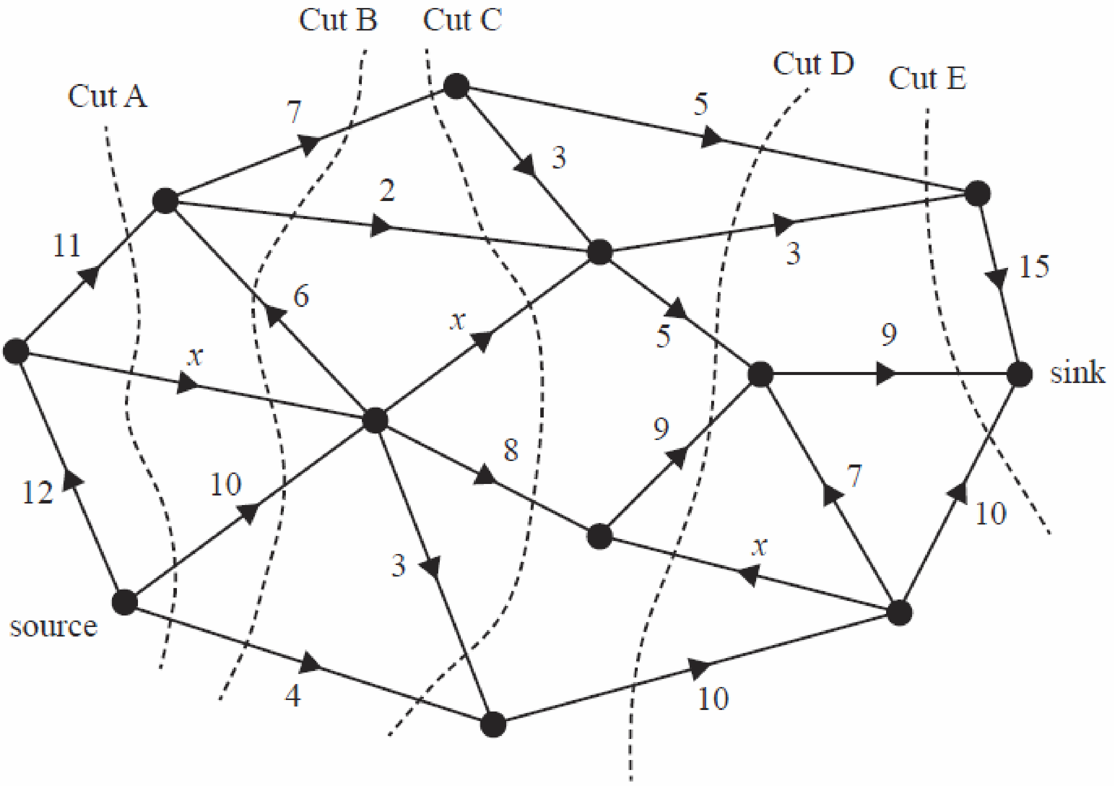
A. 10
B. 14
C. 18
D. 20
E. 22

C

The minimum cut, shown above, cuts through (6 + 10 + 2 = 18). So the maximum flow is 18
*∴ **Option C***

Question 2
(2017 Exam 1, Module 2, Qn 8)

The flow of oil through a series of pipelines, in litres per minute, is shown in the network below.



The weightings of three of the edges are labelled x.
Five cuts labelled A–E are shown on the network.
The maximum flow of oil from the source to the sink, in litres per minute, is given by the capacity of

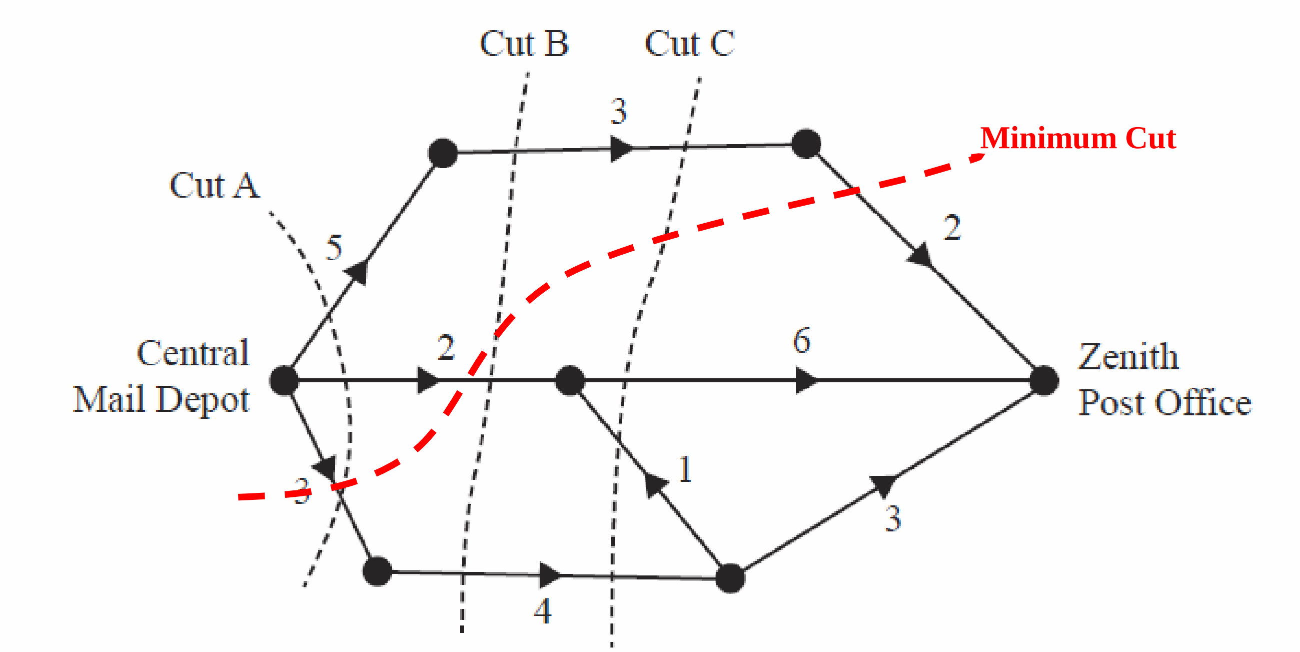
A. Cut A if x = 1
B. Cut B if x = 2
C. Cut C if x = 2
D. Cut D if x = 3
E. Cut E if x = 3

B

*Cut A (if x = 1): 11 + **1** + 10 + 4 = 26*
Cut B (if x = 2): 7 + 2 + 2 + 10 + 4 = 25
*Cut C (if x = 2): 7 + 2 + **2** + 8 + 3 + 4 = 26*
Cut D (if x = 3): 5 + 3 + 5 + 9 + 10 = 32
Cut E (if x = 3): 5 + 3 + 9 + 10 = 27

Question 3
(2018 Exam 2, Module 2, Qn 1)

The graph below shows the possible number of postal deliveries each day between the Central Mail Depot and the Zenith Post Office.
The unmarked vertices represent other depots in the region.
The weighting of each edge represents the maximum number of deliveries that can be made each day.



Part A.
Cut A, shown on the graph, has a capacity of 10.
Two other cuts are labelled as Cut B and Cut C.

i. Write down the capacity of Cut B.
Capacity of Cut B = 3 + 2 + 4 = 9

ii. Write down the capacity of Cut C.
Capacity of Cut C = 3 + 6 + 4 = 13
(NB edge of capacity 1 is traveling the wrong direction so it doesn't contribute to the cut)

Part B.
Determine the maximum number of deliveries that can be made each day from the Central Mail Depot to the Zenith Post Office.

Maximum number of deliveries = minimum cut = 7