Section 4.1.1 – Introduction to Matrices

VCAA "Dot Points"

Matrices and their applications, including:

- review of matrix arithmetic: the order of a matrix, types of matrices (row, column, square, diagonal, symmetric, triangular, zero, binary and identity), the transpose of a matrix, elementary matrix operations (sum, difference and multiplication of a scalar).
- use of matrices to represent numerical information presented in tabular form, and the use of a rule for the a_{ij}th element of a matrix to construct the matrix

---

Intro. To Matrices

A **matrix** is a rectangular **array** of numbers arranged in **rows** and **columns**. The numbers in the matrix are called the **elements** or **entry** of the matrix.

A matrix with **m rows** and **n columns** is called an (**m x n**) matrix. Matrix *A* below is an **m** by **n** matrix.

n columns

$$\textbf{m rows}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

NB: The elements within the matrix *A* are labelled *a*

Example.1

State the order of the following matrices.

Matrix A	***Matrix B***	***Matrix C***	***Matrix D***
$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 6 \end{bmatrix}$	$\begin{bmatrix} 5 & 0 & 3 \\ -1 & 4 & 7 \end{bmatrix}$	$\begin{bmatrix} 3 & 1 & 4 & 6 \\ 3 & 0 & 9 & 8 \\ 1 & 5 & 7 & 2 \end{bmatrix}$

Matrix A – Consists of **2 rows** & **1** column. ∴the order of Matrix A is (**2** × **1**)
Matrix B – Consists of **1 row** & **3** columns. ∴the order of Matrix B is (**1** × **3**)
Matrix C – Consists of **2 rows** & **3** columns. ∴the order of Matrix C is (**2** × **3**)
Matrix D – Consists of **3 rows** & **4** columns. ∴the order of Matrix D is (**3** × **4**)

The elements of the matrix *A* are referred to as a_{mn}, where ***m*** refers to the **row position** and **n** refers to the **column position**.

Example.2

Consider the following matrices.

$$A = \begin{bmatrix} 1 & 4 & 4 \\ 5 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 1 & -3 \\ 3 & -2 & 4 \end{bmatrix}$$

For both matrix A and matrix B:

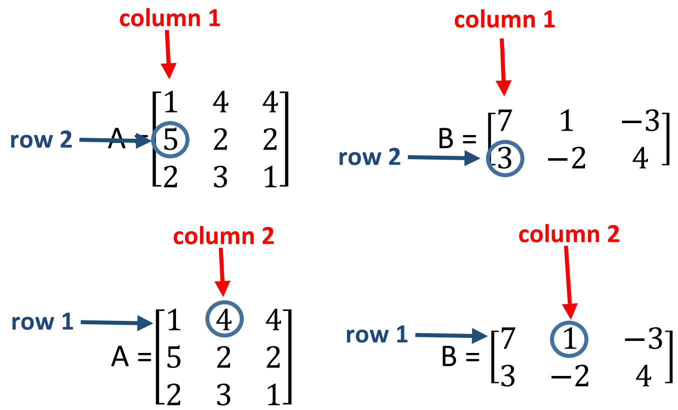
1. State the order of the matrices
2. List the elements for Matrix A & B for positions 2,1 and 1,2 within each matrix.

Solutions:

1. Matrix A is a (3 x 3) matrix
   Matrix B is a (2 x 3) matrix

2. Matrix A, position 2,1 (a_{21}) = 5
   Matrix B, position 2,1 (b_{21}) = 3

   Matrix A, position 1,2 (a_{12}) = 4
   Matrix B, position 1,2 (b_{12}) = 1



NB: In most VCAA exam questions the following notation is used:
Rows are represented by the letter *i*
Columns are represented by the letter *j*
Accordingly, any element of matrix *A* can be represented by a_{ij}.

Classification of Matrices

$\begin{bmatrix} 3 & 1 & -2 \end{bmatrix}$ **Row matrix** or row vector (1 single row)

$\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ **Column matrix** or a column vector (1 single column)

$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ **Square matrix** (equal number of rows and columns)

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **Diagonal matrix** (a square matrix with non-zero elements on the main diagonal only)

$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 5 \\ 0 & 5 & 1 \end{bmatrix}$ **Symmetric matrix** (The entries of a symmetric matrix are symmetric with respect to the main diagonal. Only square matrices can be symmetric.)

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 7 & 4 & 5 & 0 \\ 2 & 1 & 4 & 6 \end{bmatrix}$ **Triangular matrix** (a square matrix is called lower triangular if all the entries above the main diagonal are 0. Similarly, a square matrix is called upper triangular if all the entries below the main diagonal are 0.)

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **Zero matrix** (a zero matrix is a matrix consisting of all 0s. Zero matrices are sometimes also known as null matrices.)

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **Binary *matrix*** (a binary matrix is a matrix consisting of only 1s & 0s.)

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **Identity matrix** (a square matrix in which all the elements of the principal diagonal are 1s and all other elements are 0s.

Matrix A ***Matrix B***

$\begin{bmatrix} 1 & 0 \\ 7 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 7 & -3 \end{bmatrix}$ **Equal matrices** (two matrices are **equal** if they are the same order and all corresponding elements are equal, that is ***A = B.***

The Transpose of a Matrix

The **transpose** of a matrix is a new matrix whose **rows** are the **columns** of the original.
(This makes the columns of the new matrix the rows of the original).
Consider the following matrix *A*;

$A = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 0 & 4 \\ 7 & 10 & 3 \end{bmatrix}$, the transpose of matrix A would be: $A^T = \begin{bmatrix} 5 & 4 & 7 \\ 4 & 0 & 10 \\ 3 & 4 & 3 \end{bmatrix}$

NB: *A* (row 1) = A^T (column 1)
A (row 2) = A^T (column 2)
A (row 3) = A^T (column 3)

The superscript "T" means "transpose"
Non-square matrices can also be transposed. Consider the following example:

Example.3

Given that $B = \begin{bmatrix} 5 & 4 \\ 4 & 0 \\ 7 & 10 \\ -1 & 8 \end{bmatrix}$, calculate B^T

$\begin{bmatrix} 5 & 4 \\ 4 & 0 \\ 7 & 10 \\ -1 & 8 \end{bmatrix}^T$ $\begin{bmatrix} 5. & 4. & 7. & -1. \\ 4. & 0. & 10. & 8. \end{bmatrix}$

$\therefore B^T = \begin{bmatrix} 5 & 4 & 7 & -1 \\ 4 & 0 & 10 & 8 \end{bmatrix}$

Example.4

What happened when a transposed matrix is itself transposed? Let's perform a double transposition of matrix *A* above:

$$(A^T)^T = \left[\begin{bmatrix} 5 & 4 & 3 \\ 4 & 0 & 4 \\ 7 & 10 & 3 \end{bmatrix}^T\right]^T = \begin{bmatrix} 5 & 4 & 7 \\ 4 & 0 & 10 \\ 3 & 4 & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 0 & 4 \\ 7 & 10 & 3 \end{bmatrix}$$

The above calculations show that $(A^T)^T = A$

Example.5

What happened when a symmetric matrix is transposed?

$A = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 0 & 7 \\ 3 & 7 & 1 \end{bmatrix}$, the transpose of matrix A would be; $A^T = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 0 & 7 \\ 3 & 7 & 1 \end{bmatrix}$

The above calculations show that a symmetrical matrix is equal to its transpose $A^T = A$.
Note that in a symmetrical matrix $a_{ij} = a_{ji}$

Operations with Matrices

Matrices of the same **order** (or size) can be **added** and **subtracted**. This is carried out by adding and subtracting corresponding elements.

Example.6

Consider the following matrices.

$$A = \begin{bmatrix} 1 & 4 & 4 \\ 5 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -3 & 1 \\ -2 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

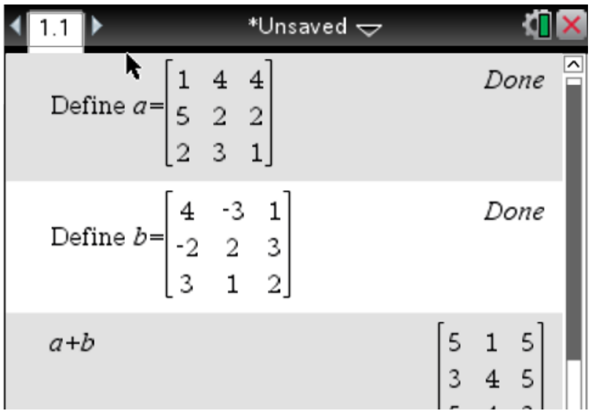
Calculate:

1. A + B
2. B – A

Solutions:

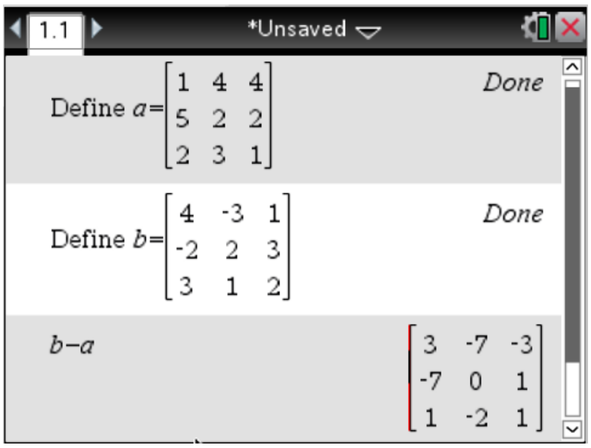
1. $$A + B = \begin{bmatrix} 1 & 4 & 4 \\ 5 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 1 \\ -2 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 & 5 \\ 3 & 4 & 5 \\ 5 & 4 & 3 \end{bmatrix}$$



2. $$B - A = \begin{bmatrix} 4 & -3 & 1 \\ -2 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 4 \\ 5 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -7 & -3 \\ -7 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$



Multiplication by a Scalar

Consider the matrix A = $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Now calculate the matrix for **4**A.

NB The number 4 is called a **scalar**.

$$4A = 4\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 \times 2 & 4 \times 1 \\ 4 \times 1 & 4 \times 3 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 12 \end{bmatrix}$$

When a matrix is multiplied by a scalar, each element of the matrix is multiplied by the scalar.

Example.7

Consider the following matrices.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$$

Calculate:

1. 2A
2. 3B
3. 2A + 3B

Solutions:

1. 2A

$$2A = 2\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 \times 2 & 2 \times 1 \\ 2 \times 1 & 2 \times 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

2. 3B

$$3B = 3\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 3 \times 1 & 3 \times 2 \\ 3 \times 4 & 3 \times 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 12 & 0 \end{bmatrix}$$

3. 2A + 3B

$$2A + 3B = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 12 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 8 \\ 14 & 6 \end{bmatrix}$$

Define $a=\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$	*Done*
Define $b=\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$	*Done*
$2\cdot a$	$\begin{bmatrix} 4. & 2. \\ 2. & 6. \end{bmatrix}$
$3\cdot b$	$\begin{bmatrix} 3. & 6. \\ 12. & 0. \end{bmatrix}$
$2\cdot a+3\cdot b$	$\begin{bmatrix} 7. & 8. \\ 14. & 6. \end{bmatrix}$

Using Element Rules (a_{ij}th) to construct the Matrix

In some circumstances you will be asked to construct a matrix following a set of "element rules", based upon the elemental references ***i*** & *j*.

Recall the elemental labelling for Matrix *A*, a_{ij}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let's consider the following example:

Example.8

The table below shows information about two matrices, A and B

Matrix	**Order**	**Rule**
A	*2×2*	$a_{ij} = i + j$
B	*2×2*	$b_{ij} = 2i - j$

The element in row *i* and column *j* of matrix *A* is a_{ij}.
The element in row *i* and column *j* of matrix *B* is b_{ij}.

<u>Construct Matrix *A*</u>
Following the rule: $a_{ij} = i + j$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (1+1) & (1+2) \\ (2+1) & (2+2) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

<u>Construct Matrix *B*</u>
Following the rule: $a_{ij} = 2i - j$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} (2 \times 1 - 1) & (2 \times 1 - 2) \\ (2 \times 2 - 1) & (2 \times 2 - 2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

<u>Construct Matrix (*A* + *B*)</u>

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix}$$

Exam Styled Questions – Multiple Choice

Question 1
(2017 Exam 1, Module 1, Qn 1)
Kai has a part-time job. Each week, he earns money and saves some of this money. The matrix below shows the amounts earned (E) and saved (S), in dollars, in each of three weeks.

$$\begin{array}{c} \\ \text{week 1} \\ \text{week 2} \\ \text{week 3} \end{array} \begin{array}{c} \begin{array}{cc} E & S \end{array} \\ \begin{bmatrix} 300 & 100 \\ 270 & 90 \\ 240 & 80 \end{bmatrix} \end{array}$$

How much did Kai save in week 2?

A. $80
B. $90
C. $100
D. $170
E. $270

B

Week 2 is located in row 2 of the above matrix and the amount saved (S) is located in column 2. So the amount saved in week 2 by Kai is located at the intersection of row 2 and column 2.
$\therefore$ ***Option B***

Question 2
(2016 Exam 1, Module 1, Qn 1)

The transpose of $\begin{bmatrix} 2 & 7 & 10 \\ 13 & 19 & 8 \end{bmatrix}$ is

A. $\begin{bmatrix} 13 & 19 & 8 \\ 2 & 7 & 10 \end{bmatrix}$

B. $\begin{bmatrix} 10 & 7 & 2 \\ 8 & 19 & 13 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 13 \\ 7 & 19 \\ 10 & 8 \end{bmatrix}$

D. $\begin{bmatrix} 13 & 2 \\ 19 & 7 \\ 8 & 10 \end{bmatrix}$

E. $\begin{bmatrix} 8 & 10 \\ 19 & 7 \\ 13 & 2 \end{bmatrix}$

C

The row of the original matrix becomes the column of the transposed matrix.
$\therefore$ ***Option C***

Question 3
(2017 Exam 1, Module 1, Qn 6)
The table below shows information about two matrices, A and B

Matrix	Order	Rule
A	3×3	$a_{ij} = 2i + j$
B	3×3	$b_{ij} = i - j$

The element in row i and column j of matrix A is a_{ij}.
The element in row i and column j of matrix B is b_{ij}.
The sum $A + B$ is

A. $\begin{bmatrix} 5 & 7 & 9 \\ 8 & 10 & 12 \\ 11 & 13 & 15 \end{bmatrix}$

B. $\begin{bmatrix} 5 & 8 & 11 \\ 7 & 10 & 13 \\ 9 & 12 & 15 \end{bmatrix}$

C. $\begin{bmatrix} 3 & 6 & 9 \\ 3 & 6 & 9 \\ 3 & 6 & 9 \end{bmatrix}$

D. $\begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 9 & 9 & 9 \end{bmatrix}$

E. $\begin{bmatrix} 3 & 6 & 3 \\ 6 & 3 & 9 \\ 3 & 9 & 3 \end{bmatrix}$

D

The elements for A+B will be $a_{ij} + b_{ij} = (2i + j) + (i - j) = 3i$

$$\therefore A + B = \begin{bmatrix} a_{11} = 3 \times 1 & a_{12} = 3 \times 1 & a_{13} = 3 \times 1 \\ a_{21} = 3 \times 2 & a_{22} = 3 \times 2 & a_{23} = 3 \times 2 \\ a_{31} = 3 \times 3 & a_{32} = 3 \times 3 & a_{33} = 3 \times 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 9 & 9 & 9 \end{bmatrix}$$

$\therefore$ ***Option D***

Question 4
(2016 Exam 1, Module 1, Qn 5)

Let $M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

The element in row i and column j of M is m_{ij}.
The elements of M are determined by the rule

A. $m_{ij} = i + j - 1$
B. $m_{ij} = 2i - j + 1$
C. $m_{ij} = 2i + j - 2$
D. $m_{ij} = i + 2j - 2$
E. $m_{ij} = i + j + 1$

C

Let's examine the top row.

	$m_{11} = 1$	$m_{12} = 2$	$m_{13} = 3$	$m_{14} = 4$
Option A	*1+1-1=1*	*1+2-1=2*	*1+3-1=3*	*1+4-1=4*
~~*Option B*~~	~~*(2×1)-1+1=2*~~			
Option C	*(2×1) + 1 – 2=1*	*(2×1) + 2 – 2=2*	*(2×1) +3 – 2=3*	*(2×1) + 4 – 2=4*
~~*Option D*~~	*1 + (2×1) – 2= 1*	~~*1 + (2×2) – 2= 3*~~		
~~*Option E*~~	~~*1 + 1 + 1=3*~~			

Only options A & C remain.
Let's examine the bottom row.

	$m_{21} = 3$	$m_{22} = 4$	$m_{23} = 5$	$m_{24} = 6$
~~*Option A*~~	~~*2 + 1 – 1=2*~~			
Option C	*(2×2) + 1 – 2=3*			

$\therefore$ ***Option C***

Question 5
(2015 Exam 1, Module 1, Qn 1)
Matrix B below shows the number of photography (P), art (A) and cooking (C) books owned by Steven (S), Trevor (T), Ursula (U), Veronica (V) and William (W).

$$B = \begin{array}{c} \\ \end{array}\begin{array}{c} P \quad A \quad C \\ \begin{bmatrix} 8 & 5 & 4 \\ 1 & 4 & 5 \\ 3 & 3 & 4 \\ 4 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ S \\ T \\ U \\ V \\ W \end{array}$$

The element in row i and column j of matrix B is b_{ij}.
The element b_{32} is the number of

A. art books owned by Trevor.
B. art books owned by Ursula.
C. art books owned by Veronica.
D. cooking books owned by Ursula.
E. cooking books owned by Trevor.

B

Element b_{32} represents Row 3 and Column 2;

- *Row 3 represents the number of books owned by Ursula*
- *Column 2 represents the number of art books owned by the people*

∴ element b_{32} represents the number of art books owned by Ursula.

∴ Option B

Question 6
(2015 Exam 1, Module 1, Qn 8)
The order of matrix X is 2×3.
The element in row i and column j of matrix X is x_{ij} and it is determined by the rule

$$x_{ij} = i - j$$

Which one of the following calculations would result in matrix X?

A. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$

A

$$X = \begin{bmatrix} a_{11} = 1 - 1 & a_{12} = 1 - 2 & a_{13} = 1 - 3 \\ a_{21} = 2 - 1 & a_{22} = 2 - 2 & a_{23} = 2 - 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

Options D & E are the wrong order
Option C is incorrect for element x_{11}
Option B is incorrect for element x_{21}
*∴ **Option A** is correct*

Question 7
(2014 Exam 1, Module 1, Qn 6)
The order of matrix X is 3×2.
The element in row i and column j of matrix X is x_{ij} and it is determined by the rule

$$x_{ij} = i + j$$

The matrix X is

A. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix}$

E. $\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$

E

$$\therefore X = \begin{bmatrix} a_{11} = 1+1 & a_{12} = 1+2 \\ a_{21} = 2+1 & a_{22} = 2+2 \\ a_{31} = 3+1 & a_{32} = 3+2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$$

$\therefore$ ***Option E*** *is correct*