Section 3.1.6 – Bell Shaped Distribution & Z Scores

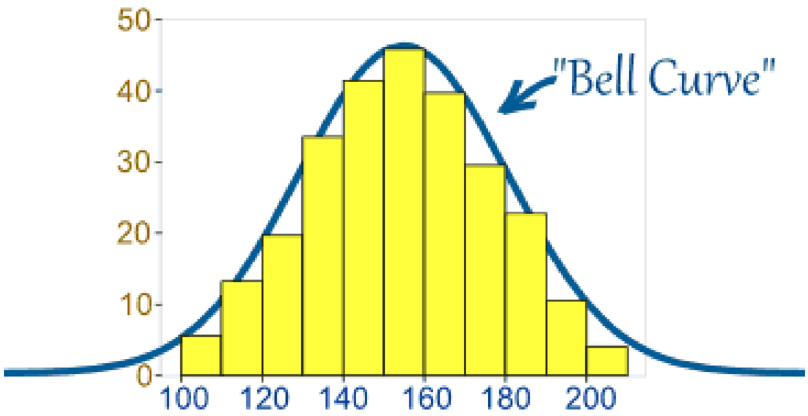
VCAA "Dot Points"

Investigating data distributions, including:

- *the normal model for bell-shaped distributions and the use of the 68–95–99.7% rule to estimate percentages and to give meaning to the standard deviation; standardised values (z-scores) and their use in comparing data values across distributions*

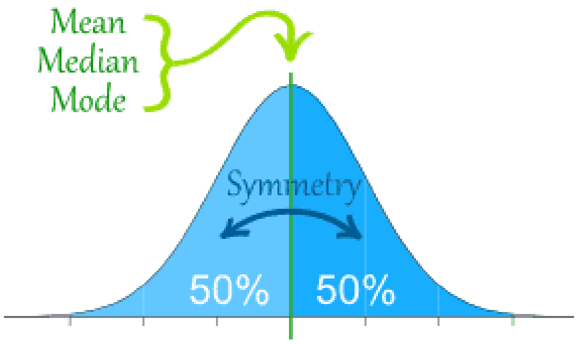
Bell Shaped Distribution

The term **bell curve** is used to describe the mathematical concept called **normal distribution**.



A normal distribution has:

- mean = median = mode
- Symmetry about the centre
- 50% of values less than the mean and 50% greater than the mean

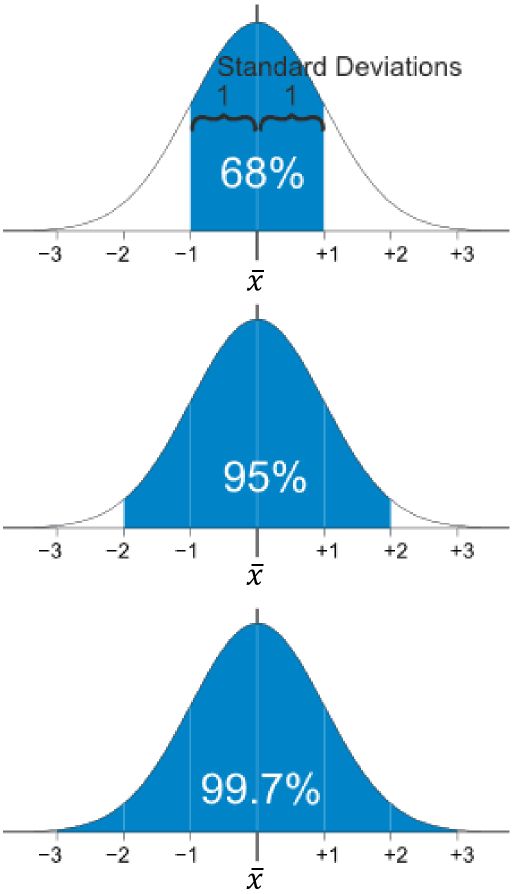


Normal distributions can be used to statistically analyse a wide range of samples, including:

- heights of people
- size of things produced by machines
- errors in measurements
- blood pressure
- marks on a test
- VCAA Study Scores

The 68 - 95 - 99.7 Rule

A **percentage distribution** within a bell curve can be made based upon the number of **standard deviations** the data spreads away from the mean. The below distribution of data is based upon 1, 2 or 3 standards deviations above and below the mean value ($\bar{x}$).

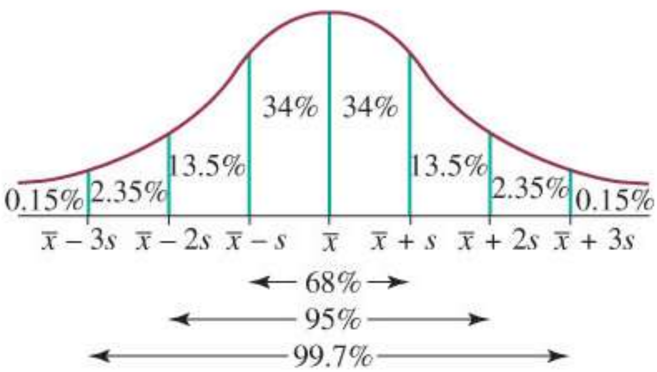


68% of values are within
1 standard deviation of the mean

95% of values are within
2 standard deviations of the mean

99.7% of values are within
3 standard deviations of the mean

The following distribution break down is a very useful tool when solving normally distributed data:



Example.1

The distribution of test scores obtained when 2500 students sit for an examination is bell-shaped with a mean of 64 and a standard deviation of 12.

From this information we can conclude that the number of these students who obtained marks between 52 and 76 is closest to:

A. 68

B. 95

C. 850

D. 1700

E. 2375

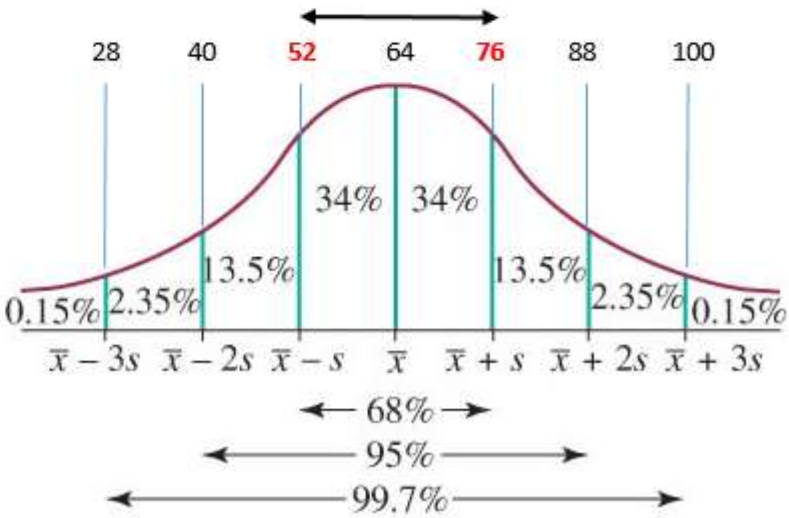
D

From the bell shaped curve it can be seen that 68% of the scores lie between the values of 52 and 76.

$\therefore$ 68% of 2500 = 1700 students

So 1700 students obtained a mark between 52 and 76.
$\therefore$ **Option D**

NB: In this example an actual number of students, rather than percentage of students, is required.



Example.2

The length of 3-month-old baby boys is approximately normally distributed with a mean of 61.1 cm and a standard deviation of 1.6 cm.

The percentage of 3-month-old baby boys with length greater than 59.5 cm is closest to

A. 5%

B. 16%

C. 68%

D. 84%

E. 95%

D

From the bell shaped curve it can be seen that baby boys of length **greate**r than **59.5 cm** represent:

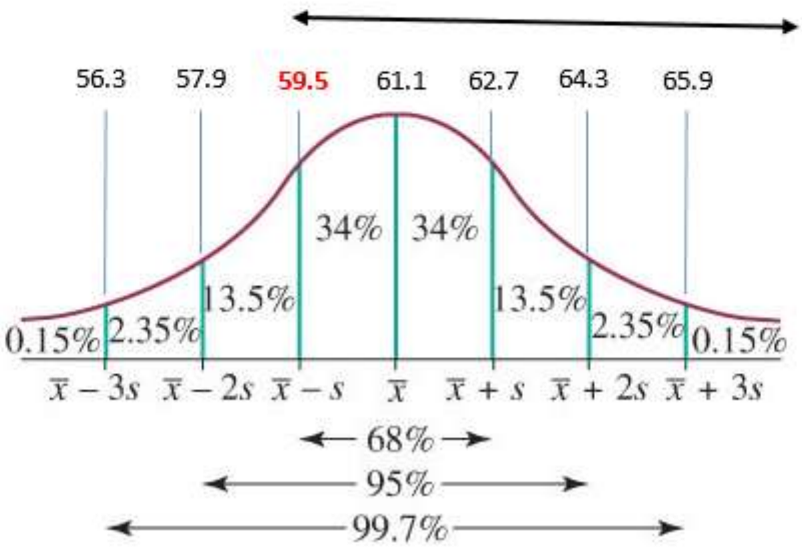
34% + 34% + 13.5% + 2.35% + 0.15% = 84%

Or alternatively;

100% - (13.5% + 2.35% + 0.15%) = 84%

∴ **option D**

NB: In this example the answer is required as a percentage.



Z-Scores

A **Z-Score** or "**standardised score**" is a numerical measure of how far an individual score is away from the mean score, within a bell shaped or normal distribution. It is measured in units of **standard deviation**.

Z-Scores are used to allow **comparison** between two different scores from two different data sets.

Z-Scores tell us whether a particular score is **equal** to the mean, **above** the mean or **below** the mean score and by how much.

Z-Score	Location of individual score
+3	Individual score is **3 standard deviations above** the mean score
+2	Individual score is **2 standard deviations above** the mean score
+1	Individual score is **1 standard deviation above** the mean score
0	Individual score is the same as the mean score
-1	Individual score is **1 standard deviation below** the mean score
-2	Individual score is **2 standard deviations below** the mean score
-3	Individual score is **3 standard deviations below** the mean score

The following equation can be used to calculate the Z-score, given the mean score ($\bar{x}$) and the standard deviation (s).

$$\boxed{z = \frac{x - \bar{x}}{s}}$$

Where z represents the z score or standardized score
x represents the actual score
$\bar{x}$ represents the mean score
s represents the standard deviation

Example.3

The mean score in an IQ test completed by a group of young footballers is 105. The standard deviation is 12.

If Imran's z-score is 2, what is his score on the IQ test?

A. 107

B. 93

C. 81

D. 117

E. 129

E

x= ?
$\bar{x}$= 105
s = 12
z = +2

This score is 2 standard deviations (s) <u>above</u> the mean score of 105.

$$z = \frac{x - \bar{x}}{s}$$

$$2 = \frac{x - 105}{12}$$

$$\text{solve}\left(2=\frac{x-105}{12},x\right) \qquad x=129$$

∴ **Option E**

Example.4

The mean number of hours per week spent studying by VCE students is 14.25 hours, with a standard deviation of 4.32.

Michelle studied for 15 hours per week. This can be represented as a z-score of;

A. 0.1736

B. -0.1736

C. 0.7495

D. -0.7495

E. 0.6620

A

z = ?
z = 15
$\bar{x}$= 14.25
s= 4.32

$$z = \frac{x - \bar{x}}{s}$$

$$z = \frac{15 - 14.25}{4.32}$$

= <u>**0.1736**</u>
∴ **Option A**

Example.5

Consider the following exam results for a Year 7 class that have been summarised according to the mean and standard deviation scores for the class.

English	$\bar{x}$ = 60 and s = 11.5
Maths	$\bar{x}$ = 72 and s = 6
Humanities	$\bar{x}$ = 56 and s = 7
Art	$\bar{x}$ = 82 and s = 4
Science	$\bar{x}$ = 46 and s = 18

John obtained 64 marks for the English exam, 80 for Maths, 61 for Humanities, 83 for Art and 60 for Science. John's best result, when compared to the class, was in;

A. English

B. Maths

C. Humanities

D. Art

E. Science

From below it can be seen that John achieved the highest z-score or standardised score in Maths.

A standardised score allows comparison between different tests.

Accordingly it can said that John performed better, relative to the rest of his class in Maths

$\therefore$ **Option B**

B

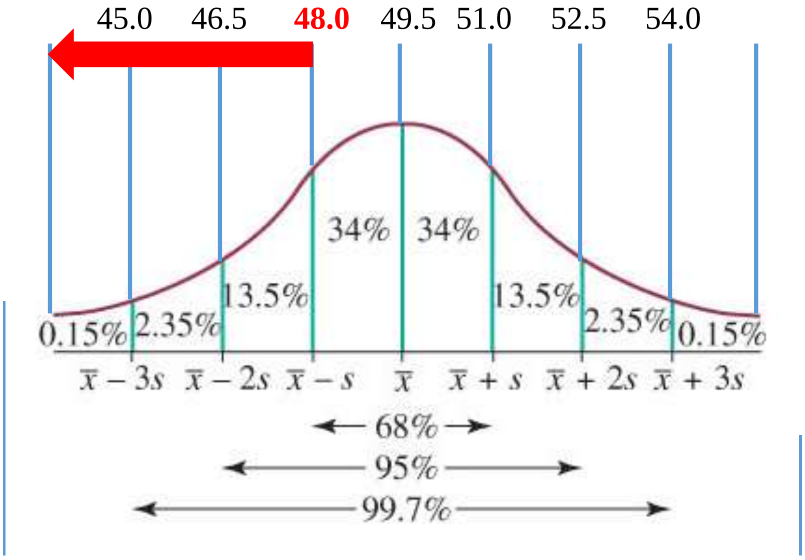
English	Maths	Humanities	Art	Science
$z = \dfrac{x - \bar{x}}{s}$	$z = \dfrac{x - \bar{x}}{s}$	$z = \dfrac{x - \bar{x}}{s}$	$z = \dfrac{x - \bar{x}}{s}$	$z = \dfrac{x - \bar{x}}{s}$
$z = \dfrac{64 - 60}{11.5}$	$z = \dfrac{80 - 72}{6}$	$z = \dfrac{61 - 56}{7}$	$z = \dfrac{83 - 82}{4}$	$z = \dfrac{60 - 46}{18}$
z = 0.3478	z = 1.3333	z = 0.7143	z = 0.25	z = 0.7778

Example.6

The head circumference (in cm) of a population of infant boys is normally distributed with a mean of 49.5 cm and a standard deviation of 1.5 cm.

Four hundred of these boys are selected at random and each boy's head circumference is measured. The number of these boys with a head circumference of less than 48.0 cm is closest to

A. 3
B. 10
C. 64
D. 272
E. 336



$\boxed{\text{C}}$

$\bar{x}$ = 49.5 cm
s = 1.5 cm

From the graph above it can be seen that (0.15% + 2.35% + 13.5% = 16%) of the population have a head circumference of less than 48.0 cm

Number of boys with head circumference < 48.0 cm = $400 \times \frac{16}{100}$
= 64 boys

∴ **Option C**

Example.7

A student obtains a mark of 56 on a test for which the mean mark is 67 and the standard deviation is 10.2. The student's standardised mark (standard z-score) is closest to

A. -1.08
B. -1.01
C. 1.01
D. 1.08
E. 49.4

$x = 56$
$\bar{x} = 67$
$s = 10.2$
Standardised mark (z-score) =?

$$z = \frac{x - \bar{x}}{s}$$
$$z = \frac{56 - 67}{10.2}$$
$= -1.08$

$\boxed{\text{A}}$

The z-score equation provides an answer of -1.08. Meaning that the mark of 56 is 1.08 standard deviations below the mean mark of the class.

∴ **Option A**

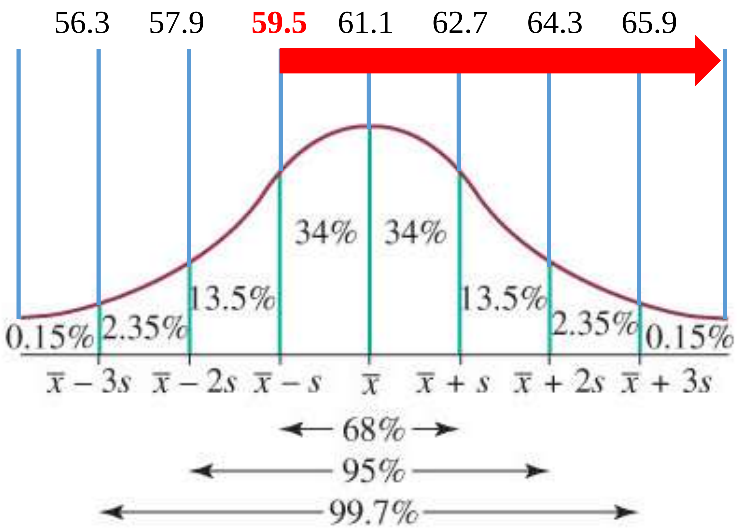
Example.8

The length of 3-month-old baby boys is approximately normally distributed with a mean of 61.1 cm and a standard deviation of 1.6 cm.
The percentage of 3-month-old baby boys with length greater than 59.5 cm is closest to

A. 5%
B. 16%
C. 68%
D. 84%
E. 95%

D

$\bar{x}$ = 61.1 cm
s = 1.6 cm



From the graph above it can be seen that (34% + 34% + 13.5% + 2.35% + 0.15% = 84%) of the population have a length greater than 59.5 cm

∴ **Option D**

The following information relates to Examples 9 and 10.

The pulse rates of a large group of 18-year-old students are approximately normally distributed with a mean of 75 beats/minute and a standard deviation of 11 beats/minute.

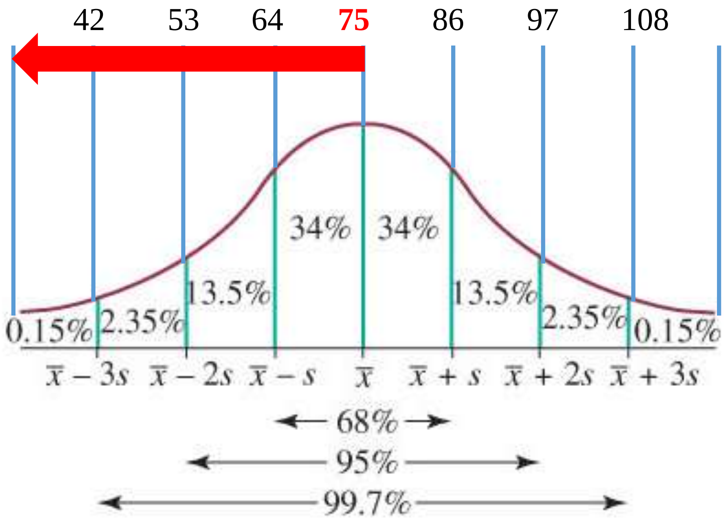
Example.9

The percentage of 18-year-old students with pulse rates less than 75 beats/minute is closest to

A. 32%
B. 50%
C. 68%
D. 84%
E. 97.5%

$\boxed{\textbf{B}}$

$\bar{x}$ = 75 beats/minute
s = 11 beats/minute



From the graph above it can be seen that (34% + 13.5% + 2.35% + 0.15% = 50%) of the population have a pulse rate less than 75 beats/minute
∴ **Option B**

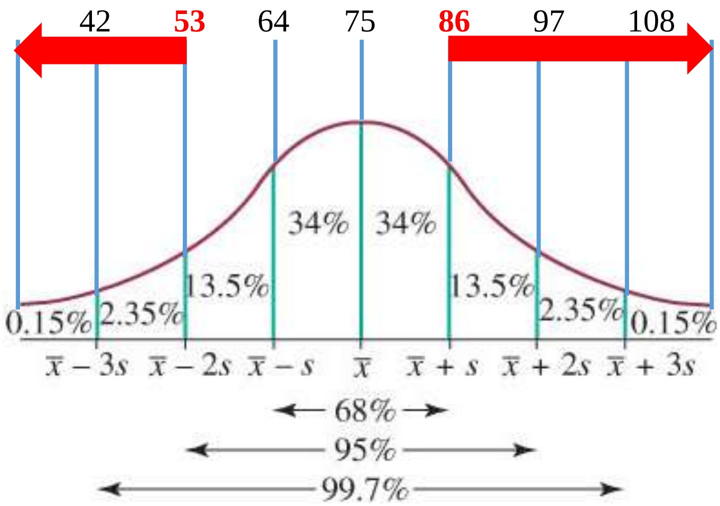
Example.10

The percentage of 18-year-old students with pulse rates less than 53 beats/minute or greater than 86 beats/minute is closest to

A. 2.5%
B. 5%
C. 16%
D. 18.5%
E. 21%

$\boxed{\textbf{D}}$

$\bar{x}$ = 75 beats/minute
s = 11 beats/minute



From the graph above it can be seen that (2.35% + 0.15% +13.5% + 2.35% + 0.15% = 18.5%) of the population have a pulse rate less than 53 beats/minute or greater than 86 beats/minute.
∴ **Option D**

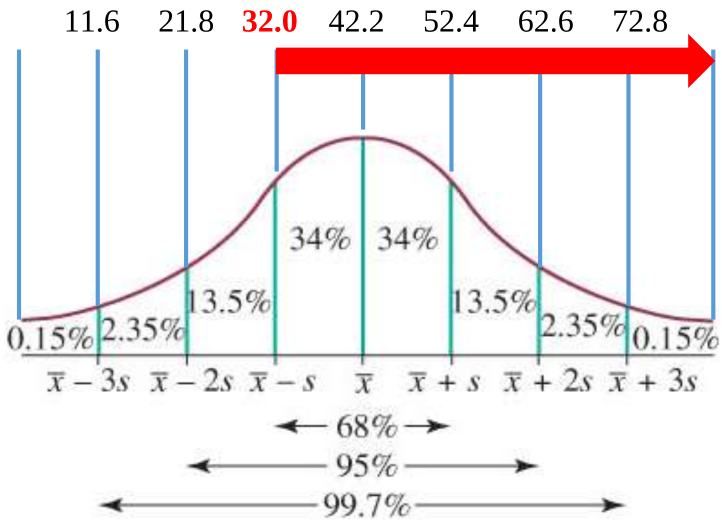
Example.11

The level of oil use in certain countries is approximately normally distributed with a mean of 42.2 units and a standard deviation of 10.2 units.
The percentage of these countries in which the level of oil use is greater than 32 units is closest to

A. 5%
B. 16%
C. 34%
D. 84%
E. 97.5%

D

$\bar{x}$ = 42.2 units
s = 10.2 units



From the graph above it can be seen that (34% + 34% + 13.5% + 2.35% + 0.15% = 84%) of the population have an oil use greater than 32 units.

∴ Option D

The following information relates to Examples 12 and 13.

The lengths of the left feet of a large sample of Year 12 students were measured and recorded. These foot lengths are approximately normally distributed with a mean of 24.2 cm and a standard deviation of 4.2 cm.

Example.12

A Year 12 student has a foot length of 23 cm.
The student's standardised foot length (standard z score) is closest to

A. –1.2
B. –0.9
C. –0.3
D. 0.3
E. 1.2

x = 23 cm
$\bar{x}$ = 24.2 cm
s = 4.2 cm
Standardised foot length (z-score) =?

$$z = \frac{x - \bar{x}}{s}$$
$$z = \frac{23 - 24.2}{4.2}$$
$$= -0.29$$

$\boxed{\text{C}}$

The z-score equation provides an answer of -0.29 (approx. -0.3). Meaning that the student's foot length of 23cm is 0.29 standard deviations below the mean foot length.

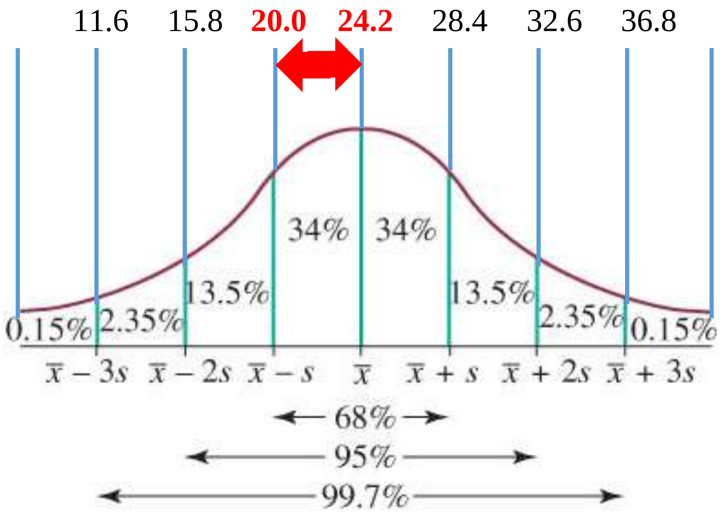
Example.13

The percentage of students with foot lengths between 20.0 and 24.2 cm is closest to

A. 16%
B. 32%
C. 34%
D. 52%
E. 68%

$\boxed{\text{C}}$

$\bar{x}$ = 24.2 cm
s = 4.2 cm



From the graph above it can be seen that 34% of the population have foot lengths between 20.0 and 24.2 cm

∴ **Option C**

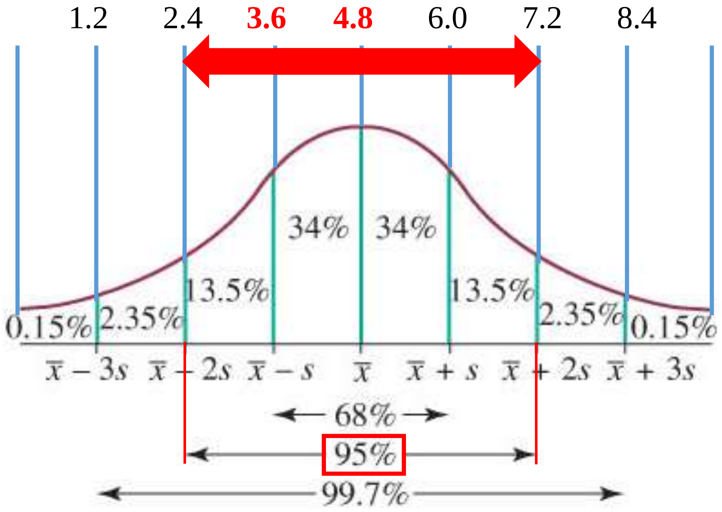
The following information relates to Examples 14 and 15.

The lengths of a type of ant is approximately normally distributed with a mean of 4.8 mm and a standard deviation of 1.2 mm

Example.14

From this information it can be concluded that around 95% of the lengths of these ants should lie between

- **A.** 2.4 mm and 6.0 mm
- **B.** 2.4 mm and 7.2 mm
- **C.** 3.6 mm and 6.0 mm
- **D.** 3.6 mm and 7.2 mm
- **E.** 4.8 mm and 7.2 mm



B

$\bar{x}$ = 4.8 mm
s = 1.2 mm

From the graph above it can be seen that 95% of the population lies between a size of 2.4 mm and 7.2 mm.

∴ **Option B**

Example.15

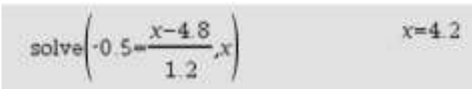
A standardised ant length of z = -0.5 corresponds to an actual ant length of

- **A.** 2.4 mm
- **B.** 3.6 mm
- **C.** 4.2 mm
- **D.** 5.4 mm
- **E.** 7.0 mm

$x = ?$
$\bar{x} = 4.8$ mm
$s = 1.2$ mm
Standardised foot length (z-score) = -0.5

$$z = \frac{x - \bar{x}}{s}$$

$$\therefore -0.5 = \frac{x - 4.8}{1.2}$$



C

From the calculations made right it can be seen that the actual ant length is 4.2 mm

∴ **Option C**

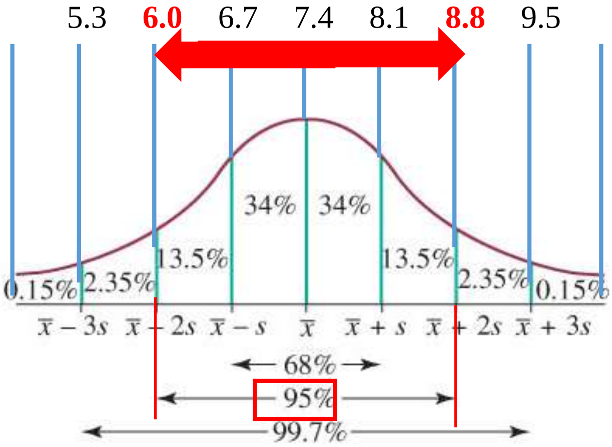
Use the following information to answer Questions 16 and 17.

The time, in hours, that each student spent sleeping on a school night was recorded for 1550 secondary-school students. The distribution of these times was found to be approximately normal with a mean of 7.4 hours and a standard deviation of 0.7 hours.

Example.16

The time that 95% of these students spent sleeping on a school night could be

A. less than 6.0 hours.
B. between 6.0 and 8.8 hours.
C. between 6.7 and 8.8 hours.
D. less than 6.0 hours or greater than 8.8 hours.
E. less than 6.7 hours or greater than 9.5 hours.



B

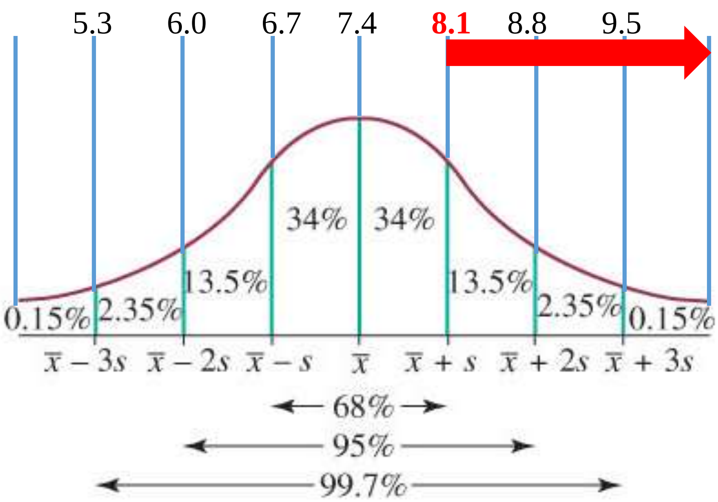
$\bar{x}$ = 7.4 hours
s = 0.7 hours

From the graph above it can be seen that 95% of the population lies between a time of 6.0 hours and 8.8 hours.
∴ **Option B**

Example.17

The number of these students who spent more than 8.1 hours sleeping on a school night was closest to

A. 16
B. 248
C. 1302
D. 1510
E. 1545



B

$\bar{x}$ = 7.4 hours
s = 0.7 hours

From the graph above it can be seen that (13.5% + 2.35% + 0.15% = 16%) of the population spent more than 8.1 hours sleeping on a school night. This represent $\frac{16}{100} \times 1550 = 248$ students
∴ **Option B**

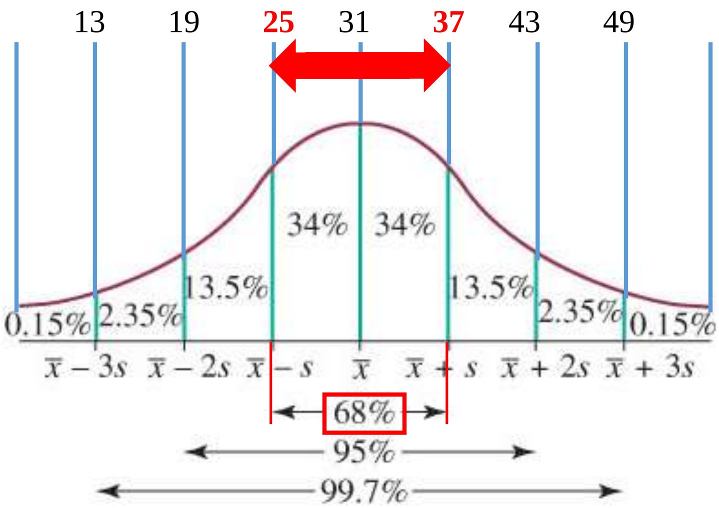
Example.18

The time spent by shoppers at a hardware store on a Saturday is approximately normally distributed with a mean of 31 minutes and a standard deviation of 6 minutes.
If 2850 shoppers are expected to visit the store on a Saturday, the number of shoppers who are expected to spend between 25 and 37 minutes in the store is closest to

A. 16
B. 68
C. 460
D. 1900
E. 2400

D

$\bar{x}$ = 31 min
s = 6 min



From the graph above it can be seen that 68% of the population lies between a time of 25 min and 37 min. In terms of the number of people this represents: $\frac{68}{100} \times 2850$ = 1938 (approx. 1900)

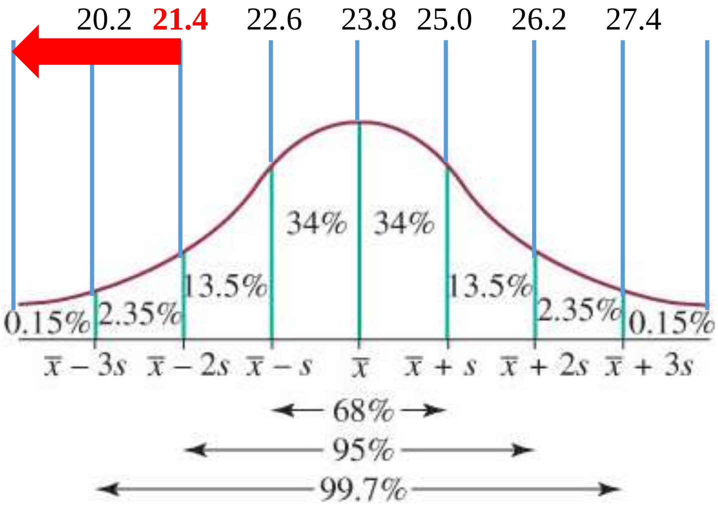
∴ Option D

Use the following information to answer Questions 19 and 20.

The foot lengths of a sample of 2400 women were approximately normally distributed with a mean of 23.8 cm and a standard deviation of 1.2 cm.

Example.19

The expected <u>number</u> of these women with foot lengths less than 21.4 cm is closest to



A. 60
B. 120
C. 384
D. 2280
E. 2340

A

$\bar{x}$ = 23.8 cm
s = 1.2 cm

From the graph above it can be seen that (2.35% + 0.15% = 2.5%) of the population have a foot length less than 21.4 cm. This represent $\frac{2.5}{100} \times 2400 = 60$ women.

∴ Option D

Example.20

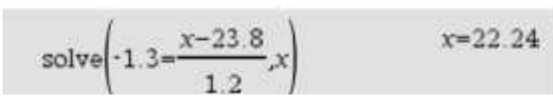
The standardised foot length of one of these women is z = –1.3
Her actual foot length, in centimetres, is closest to

A. 22.2
B. 22.7
C. 25.3
D. 25.6
E. 31.2

$x = ?$
$\bar{x} = 23.8$ cm
$s = 1.2$ cm
Standardised foot length (z-score) = -1.3

$$z = \frac{x - \bar{x}}{s}$$

$$\therefore -1.3 = \frac{x - 23.8}{1.2}$$



A

From the calculations made right it can be seen that her actual foot length is 22.24 cm (approx. 22.2 cm).

∴ Option A

Exam Styled Questions (current study design) – Multiple Choice

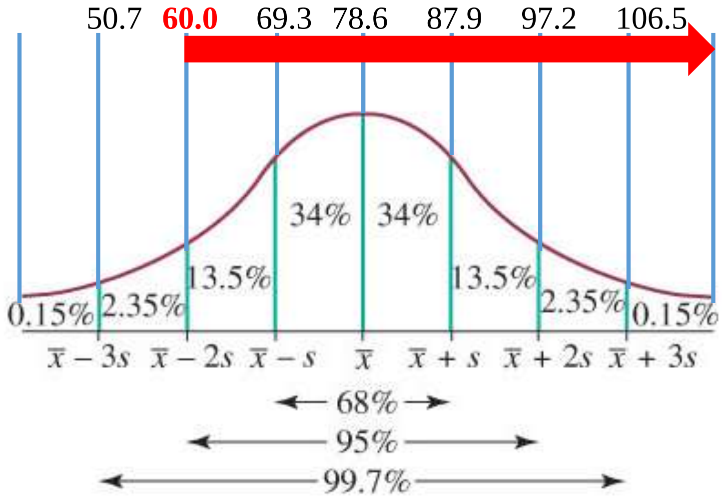
Use the following information to answer Questions 1 and 2.

The weights of male players in a basketball competition are approximately normally distributed with a mean of 78.6 kg and a standard deviation of 9.3 kg.

Question 1
(2016 Exam 1 Section A – Qn 4)

There are 456 male players in the competition.
The expected number of male players in the competition with weights above 60 kg is closest to

A. 3
B. 11
C. 23
D. 433
E. 445



$\boxed{\text{E}}$

$\bar{x}$ = 78.6 kg
s = 9.3 kg

From the graph above it can be seen that (13.5% + 34% + 34% + 13.5% + 2.35% + 0.15% = 97.5%) of the population have a weight above 60 kg. This represent $\frac{97.5}{100} \times 456$ = 444.6 male players (approx. 445 players)

$\therefore$ **Option E**

Question 2
(2016 Exam 1 Section A – Qn 5)

Brett and Sanjeeva both play in the basketball competition.
When the weights of all players in the competition are considered, Brett has a standardised weight of z = – 0.96 and Sanjeeva has a standardised weight of z = – 0.26
Which one of the following statements is not true?

A. Brett and Sanjeeva are both below the mean weight for players in the basketball competition.
B. Sanjeeva weighs more than Brett.
C. If Sanjeeva increases his weight by 2 kg, he would be above the mean weight for players in the basketball competition.
D. Brett weighs more than 68 kg.
E. More than 50% of the players in the basketball competition weigh more than Sanjeeva.

$\boxed{\text{C}}$

<u>Brett</u>
x = ?
$\bar{x}$ = 78.6 kg
s = 9.3 kg
Standardised foot length (z-score) = -0.96

$$z = \frac{x - \bar{x}}{s}$$
$$\therefore -0.96 = \frac{x - 78.6}{9.3}$$

$$\text{solve}\left(-0.96 = \frac{x - 78.6}{9.3}, x\right) \qquad x = 69.672$$

<u>Sanjeeva</u>
x = ?
$\bar{x}$ = 78.6 kg
s = 9.3 kg
Standardised foot length (z-score) = -0.26

$$z = \frac{x - \bar{x}}{s}$$
$$\therefore -0.26 = \frac{x - 78.6}{9.3}$$

$$\text{solve}\left(-0.26 = \frac{x - 78.6}{9.3}, x\right) \qquad x = 76.182$$

So Brett's weight is 69.7 kg and Sanjeeva's weight is 76.2 kg.

If Sanjeeva increased his weight by 2 kg it would become 78.2. This weight is still less than the mean value of 78.6. Therefore Option C is incorrect.

$\therefore$ **Option C**